β operator on $[0,1]$ and solution of inquality

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Abstract: At first,we propose the β operator on $[0,1]$. Next investigate relation between α and β , and obtain their properties .By applying them,we can solve inqualities in one unknown.In particular, Theorem 1.6 tell us how to develop a method for solving inquality
keywords: Fuzzy set,Operator,Inquality.

1 Operator on $[0,1]$ and their properties

We define two binary operations and unary operation,which denote by $+,*,-$ respectively

$$a+b=\max(a,b), \quad a*b=\min(a,b), \quad \overline{a}=1-a$$

Note that $a*b$ is denoted simply by ab
Now give some properties about algebra $([0,1],+,*,-)$

Theorem 1.1. $\forall a,b,c\in[0,1]$

1. $a+a=a, \quad a*a=a$ *(Idempotent)*
2. $a+b=b+a, \quad a*b=b*a$ *(commutativity)*
3. $(a+b)+c=a+(b+c), \quad (a*b)*c=a*(b*c)$ *(Associativity)*
4. $a+(a*b)=a, \quad a*(a+b)=a$ *(Absorption)*
5. $a+0=a, a+1=1, \quad a*0=0, a*1=a$ *(0-1 law)*
6. $a*(b+c)=(a*b)+(a*c), \quad a+(b*c)=(a+b)*(a+c)$ *(Distributive law)*
7. $\overline{(a+b)}=\overline{a}*\overline{b}$ *(De.Morgan law)*
8. $\overline{\overline{a}}=a$ *(Double negation law)*

Theorem 1.2. $\forall a,b,c\in L,$ *if* $a\le b,$ *then* $a+c\le b+c,$ *and* $a*c\le b*c$ *(Monotony)*

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Let $a, b \in [0,1]$, define

$$a\alpha b = \sum \{x | ax \leq b\}$$

Theorem 1.3. $a\alpha b$ *is the greatest element of the set* $\{x|ax \leq b\}$

Function $f(x) = ax$ as shown in Fig.1

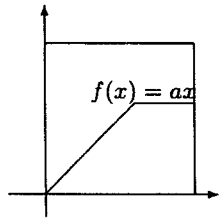
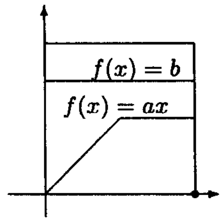


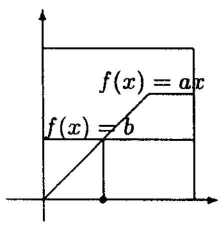
Fig.1



If $a \leq b$
Then $S = [0,1], a\alpha b = 1$
Fig.2

Consider the solution sets of the inequality $ax \leq b$:

1. When $a \leq b$: The solution set shown as Fig.2
2. When $a > b$: the solution set shown as Fig.3



If $a > b$
Then $S = [0,b], a\alpha b = b$
Fig.3

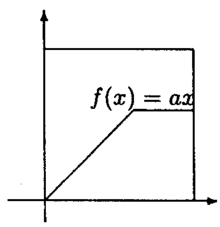


Fig.4

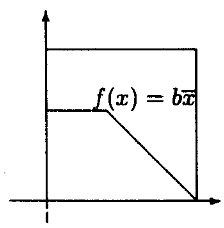


Fig.5

Hence we can define a operator α by the following

$$a\alpha b = \begin{cases} 1, & a \leq b \\ b, & a > b \end{cases}$$

Let $a, b \in [0,1]$. define

$$a\beta b = \sum \{y | ay \leq b\overline{x}\}$$

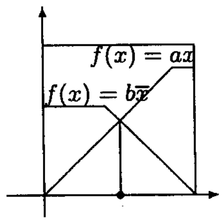
Theorem 1.4. $a\beta b$ *is the greatest element of the set* $\{y|ay \leq b\overline{y}\}$

Proof. First, $a\beta b \in \{y|ay \leq b\overline{y}\}$ Indeed:

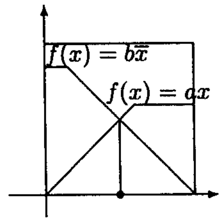
Let $Y(a,b) = \{y|ay \leq b\overline{y}\}$. For each $p \in Y(a,b)$, then $ap \leq b\overline{p}$; For any $q \in Y(a,b)$, so $aq \leq b\overline{q}$. Case 1. $p \leq q$: It is obviously that $ap \leq aq \leq b\overline{q}$. We can obtain that $ap \leq \prod_{q \in Y(a,b)} b\overline{q}$, and than $ap \leq b \prod_{q \in Y(a,b)} \overline{q}$. Again $\prod_{q \in Y(a,b)} \overline{q} = \overline{\sum_{q \in Y(a,b)} q} = \overline{a\beta b}$. So $ap \leq b\overline{a\beta b}$. It is easy to see that $\sum_{p \in Y(a,b)} ap \leq b\overline{a\beta b}$. Again $\sum_{p \in Y(a,b)} ap = a \sum_{p \in Y(a,b)} p = a(a\beta b)$. Hence $a(a\beta b) \leq b\overline{a\beta b}$. Case 2 $p > q$: It is similarly prove that $a(a\beta b) \leq b\overline{a\beta b}$.

Next,Suppose $u \in \{y|ay \leq b\overline{y}\}$. It is easy to see that $u \leq a\beta b$. □

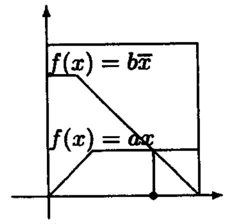
The function $f_1(x) = ax, f_2(x) = b\overline{x}$ shown as Fig.4,Fig.5 respectivly.The solution sets of inquality $ax \leq b\overline{x}$ shown as Fig.6—Fig.11,respectivly



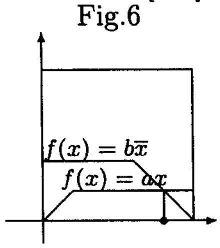
If $b \geq e, a \geq e, a \geq b$
Then $S = [0, e], a\beta b = e$
Fig.6



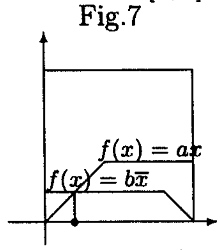
If $b \geq e, a \geq e, a < b$
Then $S = [0, e], a\beta b = e$
Fig.7



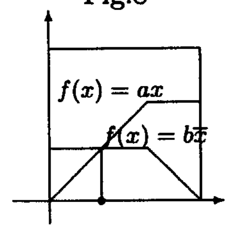
If $b \geq e, a < e$
Then $S = [0, \overline{a}], a\beta b = \overline{a}$
Fig.8



If $b < e, a \leq b$
Then $S = [0, \overline{a}], a\beta b = \overline{a}$
Fig.9



If $b < e, a > b, a < e$
Then $S = [0, b], a\beta b = b$
Fig.10



If $b < e, a \geq e, a > b$
Then $S = [0, b], a\beta b = b$
Fig.11

We can define a operator by the following

$$a\beta b = \begin{cases} e, & b \geq e, a \geq e \\ \overline{a}, & b \geq e, a < e \\ \overline{a}, & b < e, a \leq b \\ b, & b < e, a > b \end{cases}$$

It is easy to obtain the following results.

Theorem 1.5. $\forall a, b \in [0, 1]$

1. $a\alpha b \geq b$
2. *If* $a \leq b$, *then* $a\beta b \geq e$.
3. *If* $e \leq b$, *then* $a\beta b \geq e$.

The following theorem is very importment for us to solve inquality

Theorem 1.6. $\forall a, b, c \in [0, 1]$, *then the follwing statements are equivalent:*

1. $ace \leq b$
2. $\overline{a\alpha b} \leq c\alpha b$
3. $\overline{a\alpha b} \leq c\beta b$
4. $\overline{a\beta b} \leq c\alpha b$
5. $\overline{a\beta b} \leq c\beta b$

Proof. (1) $\Longrightarrow$ (2) If $ace \leq b$, then $a \leq b$ or $c \leq b$ or $e \leq b$. Case 1 $a \leq b$: since $a\alpha b = 1$, so $\overline{a\alpha b} = 0$. It is obvionsly that $\overline{a\alpha b} \leq c\alpha b$; Case 2 $c \leq b$: since $c\alpha b = 1$, hence $\overline{a\alpha b} \leq c\alpha b$; Case 3 $e \leq b$: since $a\alpha b \geq b$ and $c\alpha b \geq b$, so $\overline{a\alpha b} \leq \overline{b} \leq c\alpha b$, thus $\overline{a\alpha b} \leq c\alpha b$.

(2) $\Longrightarrow$ (1) Suppose that $ace > b$. We have $a > b$, $c > b$, and $e > b$. By definition,then $a\alpha b = b$, and $c\alpha b = b$. Again $\overline{a\alpha b} = \overline{b}$ and $e > b$, so $\overline{a\alpha b} > c\alpha b$. A contradiction.

(1) $\Longrightarrow$ (3) If $ace \leq b$, then $a \leq b$ or $c \leq b$ or $e \leq b$. case 1 $a \leq b$: since $a\alpha b = 1$, so $\overline{a\alpha b} = 0$. It is easy to see that $\overline{a\alpha b} \leq c\beta b$; Case 2 $c \leq b$: If $b < e$, then $c\beta b = \overline{c}$. Again $a\alpha b \geq b$. So $\overline{a\alpha b} \leq \overline{b} \leq \overline{c} = c\beta b$. Hence $\overline{a\alpha b} \leq c\beta b$; If $b \geq e$, then Note that $a\alpha b \geq b$, so $\overline{a\alpha b} \leq \overline{b} < e \leq c\beta b$. Hence $\overline{a\alpha b} \leq c\beta b$; Case 3 $e \leq b$: Then $a\alpha b \geq b$ and $c\beta b \geq e$, so $\overline{a\alpha b} \leq \overline{b} \leq e \leq c\beta b$. We have $\overline{a\alpha b} \leq c\beta b$.

(3) $\Longrightarrow$ (1) Suppose that $ace > b$, we obtain that $a > b, c > b, e > b$. So $a\alpha b = b$ and $c\beta b = b$. Then $\overline{a\alpha b} = \overline{b} > b = c\beta b$. It follows that $\overline{a\alpha b} > c\beta b$. A contradiction.

(1) $\Longrightarrow$ (4) If $ace \leq b$, then $a \leq b$ or $c \leq b$ or $e \leq b$. Case 1 $a \leq b$: If $b < e$, then $a\beta b = \overline{a}$. So $\overline{a\beta b} = a \leq b \leq c\alpha b$; If $b \geq e$, then $a\beta b \geq e$. So $\overline{a\beta b} \leq e \leq b \leq c\alpha b$; Case 2 $c \leq b$: we obtain that $c\alpha b = 1$. So $\overline{a\beta b} \leq c\alpha b$; Case 3 $e \leq b$: Then $a\beta b \geq e$, so $\overline{a\beta b} \leq e < b \leq c\alpha b$. Hence $\overline{a\beta b} \leq c\alpha b$.

(4) $\Longrightarrow$ (1) Suppose that $ace > b$, then $a > b, c > b, e > b$. It is easy to see that $c\alpha b = b, a\beta b = b$. So $\overline{a\beta b} = \overline{b} > b = c\alpha b$. A contradiction.

(1) $\Longrightarrow$ (5) If $ace \leq b$, then $a \leq b$ or $c \leq b$ or $e \leq b$. Case 1 $a \leq b$: If $b < e$, then $a\beta b = \overline{a}$. So $\overline{a\beta b} = a \leq b \leq c\beta b$: If $b \geq e$, then $a\beta b \geq e$ and $c\beta b \geq e$. It follows that $\overline{a\beta b} \leq e \leq c\beta b$; Case 2 $c \leq b$: We have $c\beta b \geq e$. If $b < e$, then $c\beta b = \overline{c}$. It is obviously that $\overline{c\beta b} = c \leq b \leq c\beta b$. Hence $c\beta b = \overline{c} \geq \overline{b} \geq \overline{a\beta b}$; If $b \geq e$, then $a\beta b \geq e$ and $c\beta b \geq e$. Clearly $\overline{a\beta b} \leq c\beta b$.

(5) $\Longrightarrow$ (1) Suppose that $ace > b$, Then $a > b, c > b, e > b$ It follows that $a\beta b$ and $c\beta b = b$. It is obviously that $\overline{a\beta b} = \overline{b} > b = c\beta b$. A contradiction. □

2 Inquality in One Unknown on $[0,1]$

We have discussed the properties of "α" operater and "β" operater on $[0,1]$,Now we apply them to solve Inqualities in one unknown.

Consider the following inquality.

$$a_1x + c_1\overline{x} \leq a_2x + b_2x\overline{x} + c_2\overline{x} \tag{2.1}$$

Now construct the inqualities

$$a_1x + c_1\overline{x} \leq a_2x \tag{2.11}$$

$$a_1x + c_1\overline{x} \leq b_2x\overline{x} \tag{2.12}$$

$$a_1x + c_1\overline{x} \leq c_2\overline{x} \tag{2.13}$$

Theorem 2.1. *The inquality* (2.1) *is consistent if and only if* $a_1c_1e \leq a_2+b_2+c_2$, *and*

1. *If* $a_1c_1e \leq a_2$, *then the set* $[\overline{c_1\beta a_2}, a_1\alpha a_2]$ *is the solution of the inquality (2.1)*

2. *If* $a_1c_1e \leq b_2$, *then the set* $[\overline{c_1\beta b_2}, a_1\beta b_2]$ *is the solution of the inquality(2.1)*

3. *If* $a_1c_1e \leq c_2$, *then the set* $[\overline{c_1\alpha c_2}, a_1\beta c_2]$ *is the solution of inquality(2.1)*

Proof. First Inquality (2.11) is consistent if and only if $[0, a_1\alpha a_2] \bigcap [\overline{c_1\beta a_2}, 1] \neq \emptyset$. Again $\overline{c_1\beta a_2} \leq a_1\alpha a_2$ if and only if $a_1c_1e \leq b_2$ and inquality (2.12) is consistent if and only if $a_1c_1e \leq b_2$ and inquality (2.13) is consistent if and only if $a_1c_1e \leq c_2$. Note that inquality (2.1) is consistent if and only if there is at least a inquality among inquality (2.11),(2.12) and (2.13) is consistent and the solution set ofinquality (2.1) is the union of their solution set.So inquality (2.1) is cousistent if and only if $a_1c_1e \leq a_2 + b_2 + c_2$. □

Next consider the following inquality

$$a_1x + b_1x\overline{x} + c_1\overline{x} \leq a_2x + b_2x\overline{x} + c_2\overline{x} \quad (2.2)$$

Construct two inqualities

$$(a_1 + b_1)x + c_1\overline{x} \leq a_2x + b_2x\overline{x} + c_2\overline{x} \quad (2.21)$$

$$a_1x + (b_1 + c_1)\overline{x} \leq a_2x + b_2x\overline{x} + c_2\overline{x} \quad (2.22)$$

It is obviously that

Theorem 2.2. *Inquality (2.2) is consistent if and only if* $a_1c_1e \leq a_2 + b_2 + c_2$ *and its solution set is the union of the solution set of inqualities (2.21),(2.22)*

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