ON A TYPE OF FUZZY ENTROPY

BELOSLAV RIEČAN

ABSTRACT. The notion of the Kolmogorov-Sinaj entropy of a dynamical system $(X, \mathcal{S}, P, T)$ is based on measurable set partitions $\mathcal{A} = \{A_1, \dots, A_n\}$ and their entropy $H(\mathcal{A}) = -\Sigma P(A_i) \log P(A_i)$. The notion was extended by many authors using partitions of unity, i.e., such collections $(f_1, \dots, f_n)$ of non-negative measurable functions that $\sum_{i=1}^n f_i = 1_\Omega$. In this contribution we extend this approach considering such collections $(f_1, \dots, f_n)$ of non-negative measurable functions that $\sum_{i=1}^n f_i^k = 1_\Omega$.

1. ASSUMPTIONS

There is given a dynamical system $(X, \mathcal{S}, P, T)$, where $(X, \mathcal{S}, P)$ is a probability space and $T\colon X \to X$ is a measure preserving transformation (i.e., $A \in \mathcal{S}$ implies $T^{-1}(A) \in \mathcal{S}$ and $P(A) = P(T^{-1}(A))$). By $\mathcal{F}$ we denote the family of all $\mathcal{S}$-measurable functions from X to $< 0,1 >$. Consider $k > 0$. We define a mapping $m\colon \mathcal{F} \to < 0,1 >$ by the formula $m(f) = \int_X f^k dP$.

By a Yager partition (in the paper) we mean a finite collection $f_1, \dots, f_n$ of functions from $\mathcal{F}$ such that

$$\sum_{i=1}^n f_i^k = 1_X .$$

Denote by $\mathcal{P}$ the family of all Yager partitions. If $\mathcal{A}, \mathcal{B} \in \mathcal{P}$, $\mathcal{A} = (f_1, \dots, f_n)$, $\mathcal{B} = (g_1, \dots, g_m)$, then we define

$$\mathcal{A} \vee \mathcal{B} = \{f_i \cdot g_j;\ i = 1, \dots, n,\ j = 1, \dots m\} .$$

Since

$$\sum_{i=1}^n \sum_{j=1}^m (f_i \cdot g_j)^k = \left(\sum_{i=1}^n f_i^k\right) \left(\sum_{j=1}^m g_j^k\right) = 1_X ,$$

$\mathcal{A} \vee \mathcal{B}$ is a Yager partition again. If $\mathcal{A} = (f_1, ..., f_n)$ is a partition, then we define

$$U(\mathcal{A}) = (f_1 \circ T, ..., f_n \circ T)$$

$U(\mathcal{A})$ is a Yager partition, too. Finally we define $\varphi\colon < 0, \infty) \to < 0, \infty)$ by the prescription

$$\varphi(x) = \begin{cases} -x \log x , & \text{if } x > 0 , \\ 0 & \text{if } x = 0 , \end{cases}$$

1991 *Mathematics Subject Classification.* 28E10.
Key words and phrases. Entropy, dynamical systems, fuzzy partitions, Yager conorm.
VEGA 2/6087/99

and for $\mathcal{A} \in \mathcal{P}$ the entropy

$$H(\mathcal{A}) = \sum_{i=1}^{n} \varphi(m(f_i)).$$

The term Yager partition is motivated by the Yager conorm S given by the formula $S(u,v) = min(u^k + v^k)^{1/k}, 1)$. Of course, Yager type entropy can be understood as a special case of g-entropy (see e.g. [7], [11]), since $g(u) = u^k$ implies $g^{-1}(g(u)+g(v)) = (u^k+v^k)^{1/k}$ and $g^{-1}(g(u)\cdot g(v)) = u\cdot v$. Of course, for g-entropy no analogies of Theorems 2 - 5 have been proved yet.

The complete proofs of the results presented here have been realized in [10]. Of course, we omit here some technical details (including some notions), and therefore we expect the present text to be more suitable for applications.

2. General results

Theorem 1. *Put* $a_n = H\left(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\right)$. *Then there exists* $\lim_{n\to\infty} \frac{1}{n} a_n$.

Definition. For $\mathcal{A} \in \mathcal{P}$ define

$$h(\mathcal{A}, U) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\right).$$

Theorem 2. *If* $\mathcal{A} = (f_1, \dots, f_p)$ *and* $\mathcal{B} = (\chi_{B_1}, \dots, \chi_{B_q})$ *is such that* $\sigma\left(\bigcup_{i=0}^{\infty} U^i(\mathcal{B})\right) = \mathcal{S}$, *then*

$$h(\mathcal{A}, U) \le h(\mathcal{B}, U) + \sum_{i=1}^{p} \int_X \varphi(f_i^k)\, dP.$$

As a consequence of Theorem 2 the Kolmogorov - Sinaj theorem can be obtained:

Corollary. *If* $\mathcal{A} = (\chi_{A_1}, \dots, \chi_{A_p})$ *and* $\mathcal{B} = (\chi_{B_1}, \dots, \chi_{B_q})$ *is such that* $\sigma\left(\bigcup_{i=0}^{\infty} U^i(\mathcal{B})\right) = \mathcal{S}$, *then*

$$h(\mathcal{A}, U) \le h(\mathcal{B}, U).$$

3. Hudetz correction

To obtain the Kolmogorov-Sinaj entropy one can define

$$h(T) = \sup\{h(\mathcal{A}, U);\ \mathcal{A} \in \mathcal{K}\},$$

where $\mathcal{K}$ is the family of all partitions of the form $(\chi_{A_1}, \chi_{A_2}, \dots, \chi_{A_n})$. It could be possible to define analogously the (Yager) generalized entropy as the supremum $\sup\{h(\mathcal{A}, U);\ \mathcal{A}$ consists of elements of $Q\}$, where $Q \subset \mathcal{F}$. Of course, if Q contains all constant functions, then the supremum is ∞. Therefore Hudetz suggested the following correction coefficient: for $\mathcal{A} = (f_1, \dots, f_q) \in \mathcal{P}$ put

$$\Phi(\mathcal{A}) = \sum_{i=1}^{q} \int_X \varphi(f_i^k) dP,\ H^b(\mathcal{A}) = H(\mathcal{A}) - \Phi(\mathcal{A}).$$

Theorem 3. *For any Yager partition $\mathcal{A} = (f_1, \dots, f_q)$ there exists*

$$\lim_{n\to\infty} \frac{1}{n} H^b\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\Big) = h(\mathcal{A}, U) - \sum_{j=1}^{q} \int_X \varphi(f_j^k)dP .$$

Definition. For any $\mathcal{A} \in \mathcal{P}$ define

$$h^b(\mathcal{A}, U) = \lim_{n\to\infty} \frac{1}{n} H^b\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\Big) .$$

Theorem 4. *If $\mathcal{A} = (\chi_{A_1}, \dots \chi_{A_q})$, then*

$$h^b(\mathcal{A}, U) = h(\mathcal{A}, U) .$$

Theorem 5. *Let $\mathcal{B} = \{\chi_{B_1}, \dots, \chi_{B_q}\}$ be a generator, i.e., $\sigma\Big(\bigcup_{i=1}^{\infty} U^i(\mathcal{B})\Big) = \mathcal{S}$. Then for any $\mathcal{A} = (f_1, \dots, f_p) \in \mathcal{P}$*

$$h^b(\mathcal{A}, U) \le h^b(\mathcal{B}, U) = h(\mathcal{B}, U) ,$$

hence

$$\sup\{h^b(\mathcal{A}, U);\ \mathcal{A} \in \mathcal{P}\} = h^b(\mathcal{B}, U) = h(\mathcal{B}, U) .$$

References

1. Ban, A., *Entropy of fuzzy T dynamical systems*, J. Fuzzy Math. **6** (1998), 351-362.
2. Dumitrescu, D., *Fuzzy measures and the entropy of fuzzy partitions*, J. Math. Analysis and Applications **176** (1993), 359-373.
3. Hudetz, T., *Quantum topological entropy: first step of a pedestrian approach*, Quantum Probability and Related Topics VIII (L. Accardi et al., eds.), World Scientific, Singapore, 1993, pp. 237–261.
4. Kolmogorov, A. N., *A new metric invariant of transitive dynamical systems*, Dokl. AN SSSR **119** (1958), 861–864. (In Russian)
5. Maličký, P.; Riečan, B., *On the entropy of dynamical systems*, Proc. Conf. Ergodic Theory and Related Topics II (H. Michel, ed.), Teubner, Leipzig, 1987, pp. 135–138.
6. Markechová, D., *The entropy of fuzzy dynamical systems and generators*, Fuzzy Sets and Systems **48** (1992), 351-363.
7. Mesiar, R.; Rybárik, J., *Entropy of fuzzy partitions - a general model*, Fussy Sets and Systems **99** (1998), 73-79.
8. Petrovičová, J., *On the entropy of partitions in product MV-algebras*, Soft Computing **4** (2000), 41-44.
9. Petrovičová, J., *On the entropy of dynamical systems in product MV-algebras*, Fuzzy Sets and Systems.
10. Riečan, B., *On the Yager type entropy of dynamical systems*, Manuscript, Preprint series Math, Inst. Slovak Academy of Sciences **9** (2000).
11. Riečan, B.; Neubrunn, T., *Integral, Measure and Ordering*, Kluwer, Dordrecht, Ister, Bratislava, 1997.

Mathematical Institute; Slovak Academy of Sciences; Štefánikova 49; 814 73 Bratislava; SLOVAKIA

Matej Bel University; Tajovského 40 97410 Banská Bystrica SLOVAKIA
E-mail address: riecan@mau.savba.sk