NONPROBABILISTIC MEASURES OF BIFUZZY PROBABILISTIC SETS

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ABSTRACT. In the present paper we introduce in the family of bifuzzy probabilistic sets the notions of entropy, energy and correlation as measures of fuzziness, sharpness and identification of such sets. With this we generalize the analogous notions introduced earlier for Zadeh's fuzzy sets and Hirota's probabilistic sets.

Key words: bifuzzy probabilistic set, probability measure, entropy and energy measures of a fuzzy set.

1. Introduction

The notion of a probabilistic set, introduced by K. Hirota in [9], is an alternative (in relation to fuzzy probability calculus) way of combining randomness and fuzziness. Thanks to characteristic definitions of moments, the theory of probabilistic sets has, in some respects, even greater practical prospects. Direct reasons for introducing the notion of a fuzzy probabilistic set were the premises given below which come from various areas of science (such as artificial intelligence, decision-making and pattern recognition):

1) the indefiniteness of the objects under consideration,
2) the variability of the attributes of these objects,
3) the subjectivity of the observers,
4) the evolution of knowledge and the learning of the observers.

So, let us assume that (Ω, B, P) is an arbitrary probability space. Let $(\langle 0,1\rangle, \mathrm{B}_{\langle 0,1\rangle})$ be the so-called characteristic space with the σ-field $\mathrm{B}_{\langle 0,1\rangle}$ of Borel subsets of the interval $\langle 0, 1\rangle$ and let $X \neq \phi$ be a set of some objects. Following K. Hirota [9], by a fuzzy probabilistic set A in X we mean a fuzzy set described by the so-called defining function $\mu_A: X \times \Omega \to \langle 0, 1\rangle$, written down as

$$A = \{((x,\omega), \mu_A(x,\omega)) : (x,\omega) \in X \times \Omega\},$$

the function $\mu_A(x,\cdot)$ being $(\mathcal{B}, \mathcal{B}_{\langle 0,1\rangle})$ - measurable for each $x \in X$.

In such a situation, the probability space playing the role of parameters corresponds to the subjectivity and the learning of the observers, whereas the characteristic space is connected with the indefiniteness and the variability of the attributes of the objects being examined.

In paper [6], attention was paid for the first time to the problem of measuring and comparing fuzzy probabilistic sets, with that fuzziness and sharpness measures were determined and, by this, the ideas from papers [10], [3], [4] were extended. When in 1983 K. Atanassov proposed a generalization of the notion of Zadeh's fuzzy set to a bifuzzy set [1], [2] (we deliberately do not use the name „intuitionistic" because of [13]), it was a natural conception to combine K. Atanassov's theory with that of K. Hirota. In [7] the authors introduced the basic notions and definitions and discussed the properties of bifuzzy probabilistic sets. In the present paper we point out the possibilities of defining sharpness and fuzziness measures as well as correlations of such sets, by generalizing the ideas included in [11], and discuss a number of the properties of the introduced notions.

2. Basic notions

Assuming the assumptions concerning the space $(\Omega, \mathcal{B}, P)$ and $(\langle 0,1\rangle, \mathcal{B}_{\langle 0,1\rangle})$ to be as before, we introduce [7]

DEFINITION 1. By a bifuzzy probabilistic set in the fixed space X under consideration we mean the bifuzzy set A described by a pair of defining functions $\mu_A, \nu_A : X \times \Omega \to \mathcal{B}_{\langle 0,1\rangle}$ where $\mu_A(x,\cdot)$ and $\nu_A(x,\cdot)$ are $(\mathcal{B}, \mathcal{B}_{\langle 0,1\rangle})$- measurable for each $x \in X$ and satisfy the condition

$$0 \le \mu_A(x,\cdot) + \nu_A(x,\cdot) \le 1$$

for almost all $\omega \in \Omega$. The function μ_A describes the degree of the belonging of the element x to the set A, while ν_A describes, at the same time, the degree of the non-belonging of element to A.

The family of all bifuzzy probabilistic sets in the space X is denoted by the symbol $\mathcal{P}_B(X)$.

The basic relations and operations are described by the following definition [7]:

DEFINITION 2. For $A, B \in \mathsf{P}_B(X)$, with any $x \in X$ and for almost all $\omega \in \Omega$:

1) $A \subset B \Leftrightarrow \mu_A(x,\omega) \leq \mu_B(x,\omega)$ and $\nu_A(x,\omega) \geq \nu_B(x,\omega)$,

2) $A = B \Leftrightarrow A \subset B$ and $B \subset A$,

3) $\mu_{A\cup B}(x,\omega) = \mu_A(x,\omega) \vee \mu_B(x,\omega)$ and $\nu_{A\cup B}(x,\omega) = \nu_A(x,\omega) \wedge \nu_B(x,\omega)$,

4) $\mu_{A\cap B}(x,\omega) = \mu_A(x,\omega) \wedge \mu_B(x,\omega)$ and $\nu_{A\cap B}(x,\omega) = \nu_A(x,\omega) \vee \nu_B(x,\omega)$,

5) $\mu_{A'}(x,\omega) = \nu_A(x,\omega)$ and $\nu_{B'}(x,\omega) = \mu_A(x,\omega)$

where the symbol $\vee$ stands for the operation max, and the symbol $\wedge$ for the operation min.

It can be demonstrated [7] that the family $\mathsf{P}_B(X)$ with the relation $\subset$ and the operations $\cup, \cap$ is a partially ordered set forming a Boolean pseudoalgebra. In [7] one can also find the idea of analysing the *n*-th order moments of the set $A \in \mathsf{P}_B(X)$. In this place, it is worth to notice a direct relationship between the moment of order 1 (the so-called mean value of the set *A*) and the definition of a probability of a bifuzzy event *A* [11].

3. Entropy, energy, correlation

In order to simplify the transformations, assume that $\Omega = \{\omega_1, \omega_2, \ldots, \omega_M\}$ and $X = \{x_1, x_2, \ldots, x_N\}$ are finite sets and let *A* be a bifuzzy probabilistic set in $X \times \Omega$. For a fixed $\omega_j \in \Omega$, the bifuzzy probabilistic set *A* becomes an Atanassov bifuzzy set whose entropy $d(A_j)$ is computed from the formula [5], [11].

$$d(A_j) = -K\sum_{i=1}^{N}\left[\mu_A(x_i,\omega_j)\ln\mu_A(x_i,\omega_j) + \nu_A(x_i,\omega_j)\ln\nu_A(x_i,\omega_j)\right]$$

with that $K > 0$ is some constant.

The entropy $d(A_j)$ is a measure of fuzziness of the bifuzzy set A_j [11] and, analogously to Shannon's entropy, can serve for measuring the total information on the set A_j (see also [10]). Then we introduce

DEFINITION 3. By an entropy of a bifuzzy probabilistic set *A* we mean the number

$$D(A) = \sum_{j=1}^{M} P(\omega_j) d(A_j) =$$

$$= -K\sum_{j=1}^{M} P(A_j)\left\{\sum_{i=1}^{N}\left[\mu_A(x_i,\omega_j)\cdot\ln\mu_A(x_i,\omega_j) + \nu_A(x_i,\omega_j)\cdot\ln\nu_A(x_i,\omega_j)\right]\right\}$$

The function thus defined possesses the following properties:

THEOREM 1

1) $D(A) = 0 \Leftrightarrow A$ is a set in the ordinary set-theoretical sense;

2) $D(A)$ attains its greatest value when $\mu_A \equiv \nu_A \equiv \frac{1}{2}$;

3) $D(A) = D(A')$;

4) D is a valuation;

5) $D(A) \geq D(A^s)$ where A^s is the so-called shaper version of the set A, that is, $\mu_{A^s}(x,\omega) \geq \mu_A(x,\omega) \geq \frac{1}{2}$ and $\nu_{A^s}(x,\omega) \leq \nu_A(x,\omega) \leq \frac{1}{2}$ and, at the same time, $\mu_{A^s}(x,\omega) \leq \mu_A(x,\omega) \leq \frac{1}{2}$ and $\nu_{A^s}(x,\omega) \geq \nu_A(x,\omega) \geq \frac{1}{2}$ for all $(x,\omega) \in X \times \Omega$.

The above properties follow directly from those of the auxiliary function $x \mapsto y = x\ln x$ for $x \in (0,1)$ and the so-called Shannon function

$$S(x) = \begin{cases} -x\ln x - (1-x)\ln(1-x), x \in (0,1), \\ 0 \quad x \in \{0,1\} \end{cases}$$

playing the principal role in the classical information theory, and were precisely justified in [11].

On account of Theorem 1, we may assume, according to the procedure of A. de Luca and S. Termini [10], that the entropy $D(A)$ of a bifuzzy probabilistic set A is a fuzziness (indefiniteness) of such a set.

Proceeding analogously, we introduce

DEFINITION 4. By an energy of a bifuzzy probabilistic set $A \in \mathrm{P}_B(X)$ we mean the number

$$T(A) = \sum_{j=1}^{M} P(\omega_j)\left\{\sum_{i=1}^{N}\left[\mu_A^2(x_i,\omega_j) + \nu_A^2(x_i,\omega_j)\right]\right\}$$

where $t(A_j) = \sum_{i=1}^{N}\left[\mu_A^2(x_i,\omega_j) + \nu_A^2(x_i,\omega_j)\right]$ is the energy of a bifuzzy set in the sense of Atanassov [5], [11], computed with a fixed $\omega_j \in \Omega$ for the set $A \in \mathrm{P}_B(X)$.

With that, we have

THEOREM 2

1) $T(A) = T(A')$;

2) T is a valuation;

3) $T(A)$ attains its greatest value when A is a set in the ordinary sense;

4) $T(A) \le T(A^s)$ where A^s denotes the sharper version of the set A (defined in Theorem 1).

The above four properties allow us to admit the number $T(A)$ to be a sharpness measure of a bifuzzy probabilistic set. This idea is consistent with D. Dumitrescu's conception [3] of measuring the non-fuzziness (sharpness) of a fuzzy set and is based on an alternative method for building the theory of information on the ground of the notion of 0. Onicescu's informational energy [12] instead of the entropy mentioned before.

Now, we shall introduce the notion of a correlation of bifuzzy probabilistic sets, extending with this the conceptions presented in [5], [6] and [11].

DEFINITION 5. By a correlation of sets $A, B \in \mathrm{P}_B(X)$ we mean the number

$$C(A,B) = \sum_{j=1}^{M} P(\omega_j)\left\{\sum_{i=1}^{N}\left[\mu_A(x_i,\omega_j)\cdot\mu_B(x_i,\omega_j) + \nu_A(x_i,\omega_j)\cdot\nu_B(x_i,\omega_j)\right]\right\}$$

where $c(A_j, B_j) = \sum_{i=1}^{N}\left[\mu_A(x_i,\omega_j)\cdot\mu_B(x_i,\omega_j) + \nu_A(x_i,\omega_j)\cdot\nu_B(x_i,\omega_j)\right]$ stands for a correlation of bifuzzy sets [5] A_j and B_j with a fixed $\omega_j \in \Omega$.

The correlation of the sets A and B satisfies the following evident properties:

THEOREM 3

1) $C(A,B) = C(B,A)$,

2) $C(A,A) = T(A)$.

DEFINITION 6. By a correlation coefficient of sets $A, B \in \mathrm{P}_B(X)$ we mean the number

$$K(A,B) = \frac{C(A,B)}{\sqrt{T(A)\cdot T(B)}} \quad \text{(if only } T(A), T(B) > 0\text{)}.$$

With that, we have

THEOREM 4

1) $A = B \Rightarrow K(A,B) = 1$;

2) $K(A,B) = K(B,A)$;

3) $0 \le K(A,B) \le 1$.

The considerations contained in paper [6] imply the important

THEOREM 5

$$K(A,B)=0 \Leftrightarrow |\mu_A(x_i,\omega_j)-\mu_B(x_i,\omega_j)|=1 \text{ and } |\nu_A(x_i,\omega_j)-\nu_B(x_i,\omega_j)|=1$$

for all $(x_j,\omega_j) \in X \times \Omega$.

The above condition ascertains that $K(A,B)=0$ when A and B are bifuzzy probabilistic sets with the greatest differentiation of their defining functions. In turn, Theorem 4 (assertion 1) may be interpreted as

$$K(A,B) \Leftrightarrow |\mu_A(x_i,\omega_j)-\mu_B(x_i,\omega_j)|=0 \text{ and } |\nu_A(x_i,\omega_j)-\nu_B(x_i,\omega_j)|=0,$$

which, next, means the least possible differentiation (thus equality) of the sets A and B. Therefore the correlation coefficient $K(A,B)$ expresses a degree of the identification of the sets A and B and can serve as a measure of their identity.

If $C(A,B)=0$ (equivalently, $K(A,B)=0$), then A and B are called uncorrelated bifuzzy probabilistic sets.

4. Concluding remarks

Let us observe that, in the deterministic case, the above considerations reduce to the notions of entropy, energy and correlation of bifuzzy sets in the sense of Atanassov [5], [11], introduced earlier.

It is worth emphasizing that the assumption about the finiteness of the sets X and Ω only seemingly impoverishes our considerations, but, after all, it lets the mathematical calculations be simplified to a considerable extent. Besides, in practice, finite sets (although, maybe, with a great number of elements) are far more often encountered than infinite ones. This, however, does not close the investigations of the attempts to extend the notions introduced above to infinite sets.

Let us also notice that analogous notions, although in another approach to the probabilistic situation, are discussed in paper [8].

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