Three alternative definitions of k–order additive fuzzy measures

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Abstract

k–order additivity of fuzzy measures is discussed from several points of view. Three alternative definitions are recalled or introduced. Also the generalization to k-order pseudo–additivity is given.

Key words: Additivity, fuzzy measure, k–order additivity.

1 Introduction

k–order additive discrete fuzzy measures were introduced by M. Grabisch [2] in 1996, see also [3]. For the philosophical background and discussions on properties of k–order additive fuzzy measures, including the notifications of some applications, we recommend [4]. In this note we recall only the original Grabisch's definition which works on finite universes only. To overcome the finitness restriction, we have proposed in [7] an alternative definition of k–order additive fuzzy measures independent of the cardinality of the universe we deal with. Finally, we introduce a new definition of k–order additivity which is somehow in the spirit of k–monotonicity of fuzzy measures, see, e.g., [1, 8]. All three definitions are equivalent on finite universes. The modification of all three types of definitions for the case of a pseudo–addition $\oplus$, especially for the case of $\oplus = \vee$, will be given, too.

2 Grabisch's approach

Let X be a finite non–empty set. A fuzzy measure $m : \mathcal{P}(X) \to [0, \infty]$ is a non–decreasing mapping with $m(\emptyset) = 0$. Because of the finitness of X, m is sometimes also called a discrete fuzzy measure.
The Möbius transform $M_m : \mathcal{P}(X) \to [-\infty, \infty]$ of m is defined by

$$M_m(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} m(B)\,, \tag{1}$$

where $|A \setminus B|$ is the cardinality of the relative complement of B in A. Note that the convention $+\infty + (-\infty) = +\infty$ should be used whenever that eventuality occurs.

Note also that m can be reconstructed from M_m be means of Zeta transform $m = Z_{M_m}$,

$$m(A) = \sum_{B \subset A} M_m(B) . \tag{2}$$

Definition 1 *Let m be a discrete fuzzy measure and $k \in \mathbb{N}$. Then m is called a k–order additive (discrete) fuzzy measure whenever $M_m(A) = 0$ for all $A \subset X$ with cardinality $|A| > k$.*

Note that if $k = 1$, then (1) and (2) result to

$$m(A) = \sum_{x \in A} m(\{x\}) , \tag{3}$$

i.e., m is an additive set function, and hence a classical discrete measure.
Note that each classical discrete fuzzy measure m can be represented as a linear combination of Dirac measures

$$m_x : \mathcal{P}(X) \to [0, \infty] , \qquad m_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases} , \quad x \in X ,$$

i.e.,

$$m = \sum_{x \in X} c_x m_x , \tag{4}$$

where the constant $c_x = m(\{x\}) = M_m(\{x\})$.

Similarly, each 2–order additive discrete fuzzy measure m can be written as a linear combination

$$m = \sum_{x \in X} c_x m_x + \sum_{x \neq y} d_{\{x,y\}} m_{\{x,y\}} , \tag{5}$$

where

$$m_{\{x,y\}}(A) = \begin{cases} 1 & \text{if } \{x,y\} \subset A \\ 0 & \text{else} \end{cases} , \quad |\{x,y\}| = 2,$$

and $d_{\{x,y\}} = M_m(\{x,y\})$.

Note that while c_x can be chosen as arbitrary non–negative constants, $d_{\{x,y\}}$ admit also negative values (however, the monotonicity of m restricts the freedom in the choice of values $d_{\{x,y\}}$).
Though Grabisch's Definition 1 is based on the finitness of X, it can be extended to countable universe X, too. For an additive measure m on a countable universe X (i.e., 1–order additive fuzzy measure on X) formula (4) can be directly applied. Similarly, for 2–additive fuzzy measure m on countable X we can apply (2) with $c_x \in [0, \infty]$, $d_{\{x,y\}} \in [-\infty, \infty]$ such that for any $A \subsetneqq X$ and $z \in X \setminus A$ the inequality

$$\sum_{x \in A} d_{\{x,z\}} + c_z \geq 0 \tag{6}$$

holds. Note that (6) means that $m(A) \leq m(A \cup \{z\})$. However, there is no direct way to extend Definition 1 to the case of an uncountable universe X.

3 Cartesian product–based approach

To avoid shortcuts of Grabisch's approach to the k-order additivity, we have proposed in [7] an alternative definition.

Definition 2 *Let $(X, \mathcal{A})$ be a measurable space and $m : \mathcal{A} \to [0, \infty]$ a monotone set function vanishing in the empty set, $m(\emptyset) = 0$. Let $k \in \mathbb{N}$. Then m is called a k–order additive set function on $(X, \mathcal{A})$ if there is an additive set function $m_k : \mathcal{A}^k \to [0, \infty]$ defined on the product measurable space $(X, \mathcal{A})^k$ such that for all $A \in \mathcal{A}$, $m(A) = m_k(A^k)$.*
A k–order additive set function $m : \mathcal{A} \to [0, \infty]$ is called a k–order additive fuzzy measure on $(X, \mathcal{A})$ if the corresponding additive set function m_k is weakly monotone, i.e., if $m_k(A^k) \le m_k(B^k)$ whenever $A, B \in \mathcal{A}$, $A \subset B$.

It is evident that Definition 1 is covered by Definition 2. Indeed, let X be a finite universe and let m be a k–order additive discrete fuzzy measure on X. Then it is enough to define an additive set function m_k on X^k by

$$m_k(\{(x_1, \ldots, x_k)\}) = \frac{M_m(\{x_1, \ldots, x_k\})}{|\{(y_1, \ldots, y_k)|\ \{y_1, \ldots, y_k\} = \{x_1, \ldots, x_k\}\}|},$$

to see that m is a k–order additive measure in the sense of Definition 2, too. Viceversa, if m is k–order additive in the sense of Definition 2 with corresponding additive set function m_k, we have

$$M_m(A) = m_k\left(\{(x_1, \ldots, x_k)|\ \{x_1, \ldots, x_k\} = A\}\right)$$

and hence $M_m(A) = 0$ whenever $|A| > k$.

Take, e.g., the basic 2–order additive fuzzy measure $m_{\{x,y\}}$ introduced in the previous section (obviously $m_{\{x,y\}}$ with $x \ne y$ can be defined on an arbitrary space X with $|X| > 1$). The relevant m_2 defined on $(X, \mathcal{A})^2$ (supposing $\{(x, y)\} \in \mathcal{A}^2$) can be chosen just to be the Dirac measure $m_2 = m_{\{x,y\}}$.

Note also that for any measure μ on a finite space X, the power $m = \mu^2$ is a 2–order additive discrete fuzzy measure in the sense of Definition 1. To see this not obvious fact, compute $M_m(\{x, y, z\})$ for a 3–element subset $\{x, y, z\} \subset X$:

$$\begin{aligned} M_m \quad & (\{x,y,z\}) = m(\{x,y,z\}) - (m(\{x,y\}) + m(\{x,z\}) + m(\{y,z\})) \\ + \quad & m(\{x\}) + m(\{y\}) + m(\{z\}) = (\mu(\{x\}) + \mu(\{y\}) + \mu(\{z\}))^2 \\ - \quad & \left((\mu(\{x\}) + \mu(\{y\}))^2 + (\mu(\{x\}) + \mu(\{z\}))^2 + (\mu(\{y\}) + \mu(\{z\}))^2\right) \\ + \quad & \mu^2(\{x\}) + \mu^2(\{y\}) + \mu^2(\{z\}) = 0 \end{aligned}$$

However, applying Definition 2 (on an arbitrary measurable space $(X, \mathcal{A})$), it is evident that $m = \mu^2$ is 2–order additive. Indeed, it is enough to define an additive set function $m_2 = \mu \times \mu$ on $(X, \mathcal{A})^2$.

Both approaches to the k–order additivity described in Definitions 1 and 2 are based not directly on the discussed set function m, but on some related set functions: M_m in the first case, m_k in the second one. In the next section we propose an alternative definition based directly on m only.

4 Alternative definition

The standard additive set function $m : \mathcal{A} \to [0, \infty]$ is defined by the equality

$$m(A \cup B) = m(A) + m(B) \tag{7}$$

for all $A, B \in \mathcal{A}$, $A \cap B = \emptyset$.

Inspired by Definition 2, for 2–order additive fuzzy measure m on $(X, \mathcal{A})$ with corresponding additive set function $m_2 : \mathcal{A}^2 \to [0, \infty]$ from (7) applied to m_2, we can see that for all A, B, $C \in \mathcal{A}$ pairwise disjoint we have

$$m(A \cup B \cup C) = m(A \cup B) + m(A \cup C) + m(B \cup C) - (m(A) + m(B) + m(C)) \; . \tag{8}$$

Then (8) can be taken as a characterization of the 2–additivity of m. Indeed, supposing the validity of (8), we can define

$$m_2(A \times A) = m(A)$$

and

$$m_2(A \times B) = m_2(B \times A) = \frac{m(A \cup B) - m(A) - m(B)}{2}$$

whenever A, $B \in \mathcal{A}$, $A \cap B = \emptyset$. This allows to define m_2 on an arbitrary rectangle $C \times D$ with $C \cap D = E$ by

$$\begin{aligned} m_2(C \times D) &= m_2(E \times E) + m_2((C \setminus E) \times E) \\ &+ m_2(E \times (D \setminus E)) + m_2((C \setminus E) \times (D \setminus E)) \; . \end{aligned}$$

Finally, (8) ensures the additivity of m_2.

Similarly we can discuss the case of higher orders, justifying the next alternative definition of the k–order additivity.

Definition 3 *Let m be a fuzzy measure on $(X, \mathcal{A})$ and let $k \in \mathbb{N}$. Then m is a k–order additive fuzzy measure whenever for all pairwise disjoint measurable subsets $A_1, \dots, A_{k+1} \in \mathcal{A}$ we have*

$$m\left(\bigcup_{i=1}^{k+1} A_i\right) = \sum_{I \subsetneq \{1,\dots,k+1\}} (-1)^{k-|I|} m\left(\bigcup_{i \in I} A_i\right) \; . \tag{9}$$

Note that Definition 3 is not a constructive one as the previous two definitions of the k–order additivity. On the other hand, only Definition 3 allows directly to exclude the k–order additivity (similarly as by violation of (7) we are able immediately to see the non–additivity of the discussed m). All three definitions can be modified for a pseudo–addition $\oplus$, too. However, in the case of Definition 1, first a relevant modification of the Möbius transform M_m should be done, see [6, 5]. Therefore we will continue with modifying of Definitions 2 and 3 only.

5 k–order pseudo–additive fuzzy measures

Recall that a pseudo–addition $\oplus : [0,\infty]^2 \to [0,\infty]$ is a continuous, associative and non–decreasing operation on $[0,\infty]$ with 0 as a neutral element [9, 10, 8]. Note also that the only idempotent pseudo–addition $\oplus$ is the supremum (maximum), $\oplus = \vee$.

Definition 4 *Let m be a fuzzy measure on $(X,\mathcal{A})$, $k \in \mathbb{N}$ and $\oplus$ a pseudo-addition. Then m is called a k-order $\oplus$-fuzzy measure whenever there is a weakly monotone $\oplus$-measure m_k on $(X,\mathcal{A})^k$ such that $m(A) = m_k(A^k)$, $A \in \mathcal{A}$.*

Note that m_k is $\oplus$–measure if $m_k(A \cup B) = m_k(A) \oplus m_k(B)$ whenever $A,\ B \in \mathcal{A}^k$ such that $A \cap B = \emptyset$.

An alternative definition is the next one.

Definition 5 *Let m be a fuzzy measure on $(X,\mathcal{A})$, $k \in \mathbb{N}$ and $\oplus$ a pseudo-addition. Then m is called a k-order $\oplus$-fuzzy measure whenever for any pairwise disjoint measurable subsets $A_1,\ldots,A_{k+1} \in \mathcal{A}$*

$$\bigoplus_{\substack{I \subset \{1,2,\ldots,k+1\} \\ |I| \text{ is odd}}} m\left(\bigcup_{i\in I} A_i\right) = \bigoplus_{\substack{I \subset \{1,2,\ldots,k+1\} \\ |I| \text{ is even}}} m\left(\bigcup_{i\in I} A_i\right). \tag{10}$$

Note that for the k–order maxitive fuzzy measures, i.e., when $\oplus = \vee$, (10) can be rewritten and Definition 5 modified as follows.

Definition 6 *A fuzzy measure m is a k-order maxitive fuzzy measure if and only if for any $A_1,\ldots,A_{k+1} \in \mathcal{A}$,*

$$m\left(\bigcup_{i=1}^{k+1} A_i\right) = \bigvee_{j=1}^{k+1} m\left(\bigcup_{i\neq j} A_i\right). \tag{11}$$

From (11) we can see the following: from any $(k+1)$–tuple of measurable subsets of X we can delete some member not influencing the measure of the union of discussed subsets. By induction, from any finite system $(A_i)_{i\in I}$ of measurable subsets of X with $|I| > k$, we can choose a subsystem $(A_j)_{j\in J}$, $J \subset I$, $|J| = k$ such that

$$m\left(\bigcup_{i\in I} A_i\right) = m\left(\bigcup_{j\in J} A_j\right).$$

A typical example of a 2–order maxitive fuzzy measure on a subset X of some Banach space B is the diameter

$$\operatorname{diam}(A) = \sup(\|x-y\| \mid x,y \in A).$$

Take, e.g., any finite system $(\{x_i\})_{i\in I}$ of singletons, $|I| > 2$. It is obvious that

$$\operatorname{diam}(\{x_i\}_{i\in I}) = \|x-y\| = \operatorname{diam}(\{x,y\})$$

for some $x, y \in \{x_i\}_{i\in I}$.

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