FUZZY DERIVATIVES OF FUNCTIONS GIVEN PARAMETRICALLY AND IMPLICITELY

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ABSTRACT. This is a contribution to fuzzy derivatives which have been studied in [J1 - J3] and in [K1 - K5]. Results on fuzzy derivatives of functions having multidimensionl domain [K4] and chain rule [K5] are applied to get fuzzy derivatives of functions given parametrically and implicitely. These fuzzy derivatives differ from those being studied by D. Dubois and H. Prade in [DP1 - 3].

Fuzzy derivatives of real-valued functions of one variable were introduced in [K1]. They are based on the notion of fuzzy nearness relation which (for one-dimesional case) is defined as follows (see also [K1], [J1] and [D1, D2])

Let $\mathcal{X} : \mathbb{R} \times \mathbb{R} \to [0;1]$. We say that $\mathcal{X}$ is a **fuzzy nearness relation** iff the following holds:

(1) $x\mathcal{X}z = 1 \Leftrightarrow x = z$
(2) $x\mathcal{X}z = z\mathcal{X}x$
(3) for each x, z, t if $x \leq t \leq z$ then $x\mathcal{X}t \geq x\mathcal{X}z$.
(4) for each $x \in \mathbb{R}$ $\lim_{t\to\pm\infty} x\mathcal{X}t = 0$.
(5) for all $\alpha \in]0;1]$ and all $x \in \mathbb{R}$ there exist unique $x_\alpha > x$ and $x_{-\alpha} < x$ such that $x\mathcal{X}x_\alpha = x\mathcal{X}x_{-\alpha} = \alpha$.

The 'α'-derivative of a function f at x is then the interval given by the values

$$\frac{f(x_{-\alpha}) - f(x)}{x_{-\alpha} - x}; \qquad \frac{f(x_\alpha) - f(x)}{x_\alpha - x}.$$

The notation is $\frac{df}{d\mathcal{X}}(x)$ (see also [J2, J3] and [K1, K2]).

The multidimensional version of such fuzzy derivatives was introduced in [K4]. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function continuous at a point $(x;y)$. Fix some fuzzy nearness relations $\mathcal{X}, \mathcal{Y}$, connected with the x and y variables, respectively. Denote

$$f_{-\alpha,x}'(x;y) = \frac{f(x_{-\alpha};y) - f(x;y)}{x_{-\alpha} - x}, \quad f_{\alpha,x}'(x;y) = \frac{f(x_\alpha;y) - f(x;y)}{x_\alpha - x}$$

$$f_{M_\alpha,x}'(x;y) = \max\{f_{-\alpha,x}'(x;y); f_{\alpha,x}'(x;y)\}$$

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$$f_{m_\alpha,x}{}'(x;y) = \min\{f_{-\alpha,x}{}'(x;y); f_{\alpha,x}{}'(x;y)\}$$

and similarly for the variable y.

The **partial α-derivative of f for** x at $(x;y)$ is the interval

$$\frac{\partial f}{\partial \mathcal{X}_\alpha}(x;y) = [f_{m_\alpha,x}{}'(x;y); f_{M_\alpha,x}{}'(x;y)].$$

The **partial fuzzy derivative of f for** x at $(x;y)$ is denoted by $\frac{\partial f}{\partial \mathcal{X}_x}(x;y)$ and defined by

$$\frac{\partial f}{\partial \mathcal{X}_x}(x;y) = \left[\liminf_{\alpha\to 1_-} f_{m_\alpha,x}{}'(x;y); \limsup_{\alpha\to 1_-} f_{M_\alpha,x}{}'(x;y)\right].$$

Before formulating main results yet some more notation is needed. Let $\mathcal{Z}$ be a fuzzy nearness relation. Then we denote

$$z \in_{\mathcal{Z}_\alpha} [a;b] \Leftrightarrow (z \in [a;b] \text{ or } z\mathcal{Z}a \geq \alpha \text{ or } z\mathcal{Z}b \geq \alpha).$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly monotone function. Then we denote $\varphi(\mathcal{X})$ the fuzzy nearness relation defined by

$$\tilde{t}\ \varphi(\mathcal{X})\ t = \varphi^{-1}(\tilde{t})\ \mathcal{X}\ \varphi^{-1}(t).$$

By the addition and multiplication of intervals (subsets of $]-\infty;\infty[$) we mean

$$[a_1;a_2] * [b_1;b_2] = [\min\{a_1 * b_1; a_1 * b_2; a_2 * b_1; a_2 * b_2\};\\ \max\{a_1 * b_1; a_1 * b_2; a_2 * b_1; a_2 * b_2\}],$$

where $*$ is either addition or multiplication.

Theorem 1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous, strictly monotone function and $\psi : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$ be continuos functions. Fix some $t \in \mathbb{R}$. Denote $\mathcal{T}$, $\mathcal{X}$, $\mathcal{Y}$ some fuzzy nearness relations, connected with the variables t, x, y, respectively and further denote*

$$x = \varphi(t), \quad y = \psi(t), \quad f(t) = F(\varphi(t);\psi(t)).$$

Assume the fuzzy derivatives $\frac{\partial F}{\partial \mathcal{X}}(x;y)$, $\frac{d\varphi}{d\mathcal{T}}(t)$, $\frac{\partial F}{\partial \mathcal{Y}}(x;y)$, $\frac{d\psi}{d\mathcal{T}}(t)$ to be finite (i.e. to be subsets of $]-\infty;\infty[$). Then if there exists a fuzzy nearness relation $\mathcal{Z}$ such that

$$F_{-\alpha,y}{}'(\varphi(t_{-\alpha});y) \in_{\mathcal{Z}_\alpha} \frac{\partial F}{\partial \mathcal{Y}_\alpha}(x;y) \quad \& \quad F_{\alpha,y}{}'(\varphi(t_\alpha);y) \in_{\mathcal{Z}_\alpha} \frac{\partial F}{\partial \mathcal{Y}_\alpha}(x;y),$$

then

$$\frac{df}{d\mathcal{T}}(t) \subseteq \frac{\partial F}{\partial \mathcal{X}}(x;y)\frac{d\varphi}{d\mathcal{T}}(t) + \frac{\partial F}{\partial \mathcal{Y}}(x;y)\frac{d\psi}{d\mathcal{T}}(t).$$

Theorem 2. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that*

$$F(u; v) = c$$

defines a function $\psi : \mathbb{R} \to \mathbb{R}$ ($\psi(x) = y$). Denote $\mathcal{X}$ and $\mathcal{Y}$ some fuzzy nearness relations connected with the variables x and y, respectively. Let the fuzzy derivatives $\frac{\partial F}{\partial \mathcal{X}}(x; y)$, $\frac{\partial F}{\partial \mathcal{Y}}(x; y)$, $\frac{d\psi}{d\mathcal{X}}(x)$ to be finite (i.e. to be subsets of $]-\infty; \infty[$). Assume that there is a fuzzy nearness relation $\mathcal{Z}$ such that

$$F_{-\alpha,y}{}'(x_{-\alpha}; y) \in_{\mathcal{Z}_\alpha} \frac{\partial F}{\partial \mathcal{Y}_\alpha}(x; y) \quad \& \quad F_{\alpha,y}{}'(x_\alpha; y) \in_{\mathcal{Z}_\alpha} \frac{\partial F}{\partial \mathcal{Y}_\alpha}(x; y).$$

Then

$$-\frac{\partial F}{\partial \mathcal{X}}(x; y) \subseteq \frac{\partial F}{\partial \mathcal{Y}}(x; y) \frac{d\psi}{d\mathcal{X}}(x).$$

Corollary to Theorem 2. *Assume all the conditions from Theorem 2 hold. Denote $\frac{\partial F}{\partial \mathcal{X}}(x; y) = [a; b]$ and $\frac{\partial F}{\partial \mathcal{Y}}(x; y) = [c; d]$ Then*

(a) *if $\frac{\partial F}{\partial \mathcal{X}}(x; y) \subset]-\infty; 0[$ and $\frac{\partial F}{\partial \mathcal{Y}}(x; y) \subset]0; \infty[$, then*

$$\frac{d\psi}{d\mathcal{X}}(x) \supseteq \left[-\frac{b}{c}; -\frac{a}{d}\right]$$

(b) *if $\frac{\partial F}{\partial \mathcal{X}}(x; y) \subset]0; \infty[$ and $\frac{\partial F}{\partial \mathcal{Y}}(x; y) \subset]-\infty; 0[$, then*

$$\frac{d\psi}{d\mathcal{X}}(x) \supseteq \left[-\frac{a}{d}; -\frac{b}{c}\right]$$

(c) *if $\frac{\partial F}{\partial \mathcal{X}}(x; y) \subset]0; \infty[$ and $\frac{\partial F}{\partial \mathcal{Y}}(x; y) \subset]0; \infty[$, then*

$$\frac{d\psi}{d\mathcal{X}}(x) \supseteq \left[-\frac{b}{d}; -\frac{a}{c}\right]$$

(d) *if $\frac{\partial F}{\partial \mathcal{X}}(x; y) \subset]-\infty; 0[$ and $\frac{\partial F}{\partial \mathcal{Y}}(x; y) \subset]-\infty; 0[$, then*

$$\frac{d\psi}{d\mathcal{X}}(x) \supseteq \left[-\frac{a}{c}; -\frac{b}{d}\right]$$

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