INDISTINGUISHABILITY AND NEARNESS

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ABSTRACT. Fuzzy equivalence relations play a significant role in fuzzy set theory, analogical to the role of crisp equivalence relations in classical theory. On the other hand, nearness is a natural fuzzification of distance. Both these binary fuzzy relations represent a possible approach to indistinguishability and proximity of elements in a universe. The main purpose of this paper is to discuss their mutual correspondence on an arbitrary universe.

1. PRELIMINARIES

This paper is intended to discuss two types of binary fuzzy relations on a universe X, namely fuzzy equivalences and nearnesses and their mutual connection. This topic was studied already in [3], but only for the real case, provided that $X = \mathbf{R}$.

First recall some basic notions in the fuzzy theory and triangular norms theory([7]):

A triangular norm (briefly, t-norm) $\mathcal{T}$ is a binary operator on $[0,1]$ that is nondecreasing, associative, commutative and such that $\mathcal{T}(1,x) = x$, for each $x \in [0,1]$.

A t-norm $\mathcal{T}_1$ is said to be weaker than a t-norm $\mathcal{T}_2$ if $\mathcal{T}_1(x,y) \le \mathcal{T}_2(x,y)$ for each $x, y \in [0,1]$.

A strictly decreasing mapping $f\colon [0,1] \to [0,\infty]$ with range relatively closed under addition, such that $f(1) = 0$, is called an additive generator.

If f is an additive generator then the mapping $f^{(-1)}\colon [0,\infty] \to [0,1]$ defined by:

$$f^{(-1)}(x) = \inf\{t :\ t \in [0,1] \wedge f(t) \le x\}$$

is called the pseudo-inverse of f.

If f is an additive generator, then the mapping $\mathcal{T}\colon [0,1]^2 \to [0,1]$ defined by

$$\mathcal{T}(x,y) = f^{(-1)}\big(f(x) + f(y)\big)$$

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is a t-norm, which is Archimedean (see for example [1]). Moreover, $\mathcal{T}$ is continuous if and only if f is continuous ([1]).

Fuzzy equivalence relations were introduced by L.A.Zadeh in [8] under the name similarity relations for a special case and lately generalized:

Definition 1. Let $\mathcal{T}$ be a t-norm. A binary fuzzy relation E on a universe X is called a *$\mathcal{T}$-equivalence* on X if and only if it is reflexive, symmetric and $\mathcal{T}$-transitive, it means, if and only if for any $x, y, z \in X$:

(E1) $E(x,x) = 1$,
(E2) $E(x,y) = E(y,x)$,
(E3) $\mathcal{T}(E(x,y), E(y,z)) \leq E(x,z)$.

If moreover $E(x,y) = 1$ implies $x = y$, E is called a *$\mathcal{T}$-equality*.

Evidently, if a binary fuzzy relation E is a $\mathcal{T}$-equivalence on X, then it is also a $\mathcal{T}^*$-equivalence for any weaker t-norm $\mathcal{T}^*$.

In the papers [1] and [2] relations between $\mathcal{T}$-equivalences and pseudometrics and $\mathcal{T}$-equalities and metrics on a universe X are discussed. These papers continue and in fact complete recent works concerning this topic.

In the paper [1] there is shown that in the construction of pseudometrics from $\mathcal{T}$-equivalences additive generators play a substantial role and conversely, in the construction of $\mathcal{T}$-equivalences this role is played by pseudo-inverses of additive generators. Of the greatest importance for us is the following

Theorem 1. ([1; Proposition 7]) *Let d be a pseudometric on a universe X and $\mathcal{T}$ be a continuous Archimedean t-norm with additive generator f. Then the binary fuzzy relation $E = f^{(-1)} \circ d$ on X, i.e.*

$$E(x,y) = f^{(-1)}\left(d(x,y)\right),$$

is a $\mathcal{T}$-equivalence on X.

Analogical results are in [2] made for $\mathcal{T}$-equalities and metrics. There is also proved that foregoing theorem is valid, if the pseudometric is replaced by a metric, and $\mathcal{T}$-equivalence by a $\mathcal{T}$-equality. These results correspond with our intuitive comprehension of $\mathcal{T}$-equivalences as an expression of a degree of similarity, indistinguishability, or, in a sence, "proximity" for elements in a universe.

Another approach to the fuzzification of indistinguishability , convergence, continuity and relevant notions is based on the concept of a nearness. Investigations in this direction are for instance in [3],[4],[5] and [6].

In [3] a binary fuzzy relation N on $\mathbf{R}$ is called a shift-invariant nearness if there exists such a non-increasing function $b\colon [0,\infty] \to [0,1]$ that $b(0) = 1$, $\lim_{x\to\infty} b(x) = 0$ and

$$N(x,y) = b(|x-y|) \text{ for each } x, y \in \mathbf{R}.$$

The function b is said to be a nearness-generating function.

There it is proved that just such a binary fuzzy relation on $\mathbf{R}$ is appropriate as a natural fuzzification of standard metric in $\mathbf{R}$, which is compatible also with algebraical and lattice structure of real numbers. Correspondence between $\mathcal{T}$-equalities and nearnesses in real case follows from

Theorem 2. *Let N be a shift-invariant nearness and let its nearness-generating function b be continuous and decreasing on an interval $[0,a], a \in (0,\infty]$ and let $f(a)=0$.*

Then b is a pseudoinverse of a continuous Archimedean t-norm $\mathcal{T}^*$ and N is a $\mathcal{T}$-equality for any t-norm $\mathcal{T}$ weaker than $\mathcal{T}^*$.

The statement is a simple consequence of Theorem 1 and Theorem 2 in [3]. Any "reasonable" nearness on the set of all real numbers should be evidently compatible with the structure of real axis. But if we consider nearness on an arbitrary universe X, without any structure, we are no more restricted by any additional requirements and conditions and a nearness on X should comply only with our intuitive idea of a nearness.

2. Nearness and $\mathcal{T}$ -equivalence

In [4] the nearness N on a universe X was defined as follows.

Definition 2.

Let X be a set. A binary fuzzy relation N on X is called a nearness on X, if:

(N1) $N(x,x)=1$, for each $x \in X$

(N2) $N(x,y)=N(y,x)$, for each $x,y \in X$

(N3) For each $\epsilon > 0$ there exists $\delta < 1$ such that

$$N(x,y) > \delta \Longrightarrow |N(x,z) - N(y,z)| < \epsilon, \text{ for each } x,y,z \in X.$$

The properties (N1) and (N2), it means the reflexivity and the symmetricity of N are immediate. The property (N3) substitutes, in a sence, the triangular inequality and it has the following meaning: If two points x and y are sufficiently near one another, then the difference of their nearnesses to any other point z is arbitrarily small.

In addition, N is called separated, if

(N1') $N(x,y)=1 \Longleftrightarrow x=y,$ for each $x,y \in X$.

Remark 1. If R is a reflexive and symmetric binary fuzzy relation on X, satisfying (N1'), such that the value 1 is an isolated point of its range, it means that

$$a = \sup\{b : b \in [0,1) \text{ and there exist } x,y \in X : R(x,y)=b\} < 1,$$

then it is evident that R satisfies (N3), hence it is a nearness.

(For any $\epsilon > 0$ it is sufficient to take an arbitrary $\delta \in (a,1)$.)

Therefore, if for example range of R is a finite set, R is a nearness.

Example 1. Let X be a universe consisting of three elements: $X = \{x, y, z\}$. Let R be such a reflexive and symmetrical binary fuzzy relation on X that

$$R(x,y) = 1,\ R(x,z) = \frac{1}{2},\ R(y,z) = \frac{2}{3}.$$

It is obvious, that R is not a nearness. Thus the property (N1') cannot be dropped.

A trivial verification shows that the property (N3) implies the property

(N3') For each $x, y, z \in X$:

$$N(x,y) = 1 \Longrightarrow N(x,z) = N(y,z)$$

Hence, if N doesn't distinguish two elements from an universe, if they have the maximal possible degree of nearness, then their nearness to any other element of the universe is of the same degree.

The following example shows that the opposite is not true, the property (N3') is weaker than (N3) and also that a shift-invariant nearness needn't be the nearness defined above.

Example 2. Consider a shift-invariant nearness

$$N(x,y) = b(|x-y|) \text{ for each } x, y \in \mathbf{R},$$

where the nearness-generating function b is defined by

$$b(x) = \begin{cases} 1 - \frac{x}{10}, & \text{for } x < 5, \\ 0, & \text{for } x \geq 5. \end{cases}$$

It is easily checked, that for any $\delta < 1$ there is a number n_0 such that for each $n \in \mathbf{N}$, $n > n_0$,

$$N(\frac{1}{n}, \frac{-1}{n}) = b(\frac{2}{n}) = \frac{5n-1}{5n} > \delta$$

and simultaneously

$$|N(5, \frac{1}{n}) - N(5, \frac{-1}{n})| = \frac{1}{2} + \frac{1}{10n} > \frac{1}{2}.$$

It means, that N does not satisfy (N3), hence it is not the nearness in the sence of Definition 2. But it is easily seen, that the propery (N3') is fulfilled.

Theorem 3. *Let a fuzzy equivalence E on a universe X be expressed as follows: $E = b \circ d$ on X, i.e.*

$$E(x,y) = b\big(d(x,y)\big),$$

where d is a pseudometric on X and $b : [0, \infty] \to [0, 1]$ is such a continuous and decreasing function that $b(0) = 1$, $\lim_{x \to \infty} b(x) = 0$.

Then E is a nearness on X. If moreover d is a metric, then the nearness is separated.

Proof. Properties (N1) and (N2) are immediate consequences of corresponding properties of any pseudometric. It remains to prove, that also the property (N3) is fulfilled.

We first show, that the function b is uniformly continuous. Suppose on the contrary that there exists $\varepsilon > 0$ such that for any natural number n there exist nonnegative real numbers x_n, y_n such that $|x_n - y_n| < \frac{1}{n}$ but $|b(x_n) - b(y_n)| \geq \varepsilon$.

If there exists such a positive number K that all x_n, y_n are less than K, we obtain a cotradiction, because b on the compact $[0, K]$ is uniformly continuous. It follows, that there must exist subsequences $\{x_{n_k}\}$, $\{y_{n_k}\}$, both approaching infinity, while sequences $\{b(x_{n_k})\}, \{b(y_{n_k})\}$ both converge to 0, which contradicts to assumption.

We proceed to show that for each $\varepsilon > 0$ there exists $\delta < 1$ such that

$$b(d(x,y)) > \delta \implies |b(d(x,y)) - b(d(y,z)| < \varepsilon, \text{ for each } x, y, z \in X.$$

Let $\varepsilon > 0$, then there exists $\delta_0 > 0$ such that

$$|d(x,z) - d(y,z)| < \delta_0 \implies |b(d(x,z) - b(d(y,z))| < \varepsilon.$$

Put $\delta = b(\delta_0)$. Clearly $\delta < 1$.

For arbitrary $x, y, z \in X$ if $b(d(x,y)) > \delta$ then $d(x,y) < \delta_0$ and triangular inequality of d and uniform continuity of b imply

$$|d(x,z) - d(y,z)| \leq d(x,y) \implies |b(d(x,z)) - b(d(y,z))| < \varepsilon.$$

From the last theorem, Theorem 1 and Proposition 6 in [1] it follows

Corollary 1. *Let X be a universe and $\mathcal{T}$ be a continuous Archimedean t-norm with additive generator. Then each $\mathcal{T}$-equivalence on X is a nearness.*

As the following example shows, construction of $\mathcal{T}$-equivalences introduced in Theorem 1 fails if we omitt the requirement of continuity

Example 3. Consider a shift-invariant nearness

$$N(x,y) = b(|x-y|) \text{ for each } x, y \in \mathbf{R},$$

where the nearness-generating function b is defined by

$$b(x) = \begin{cases} 1, & \text{for } x < 1, \\ 2 - x, & \text{for } x \in [1,2], \\ 0, & \text{else.} \end{cases}$$

The function b is pseudoinverse of the weakest t-norm $\mathcal{T}_W$, defined by

$$\mathcal{T}_W(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{else.} \end{cases}$$

Since $\mathcal{T}_W$ and its additive generator are not continuous, N is neither $\mathcal{T}_W$-transitive:

$$\mathcal{T}_W(N(0,1), N(1,\tfrac{3}{2})) = 1 > \frac{1}{2} = N(0,\tfrac{3}{2}).$$

It follows, that the binary fuzzy relation N is not $\mathcal{T}$-equivalence for any t-norm.

The binary fuzzy relation N satisfies neither the property (N3'): $N(0,1) = 1$, but $N(0,\frac{3}{2}) = \frac{1}{2} \neq N(1,\frac{3}{2})$.

This fact folows also from the next theorem.

Theorem 4. *Reflexive and symmetrical binary fuzzy relation possesses the property (N3') if and only if it is $\mathcal{T}_W$ -transitive.*

Proof. Let R be reflexive, symmetrical and $\mathcal{T}_W$-transitive on a universe X. Suppose $x, y \in X$ are such that $R(x,y) = 1$. Then from the definitions of $\mathcal{T}_W$ and $\mathcal{T}_W$-transitivity it follows for any $z \in X$:

$$\mathcal{T}_W(R(x,y), R(y,z)) = R(y,z) \leq R(x,z)$$

and simultaneously

$$\mathcal{T}_W(R(y,x), R(x,z)) = R(x,z) \leq R(y,z).$$

Combining these inequalities we have $R(x,z) = R(y,z)$.

Now let R be reflexive, symmetrical and satisfying (N3') and let $x, y, z \in X$ be arbitrary. From the definitions of $\mathcal{T}_W$ and (N3') it follows:

If $\max(R(x,y), R(y,z)) = 1$ then

$$R(x,z) = \min(R(x,y), R(y,z)) = \mathcal{T}_W(R(x,y), R(y,z))$$

and if $\max(R(x,y), R(y,z)) < 1$ then

$$\mathcal{T}_W(R(x,y), R(y,z)) = 0 \leq R(x,z).$$

Corollary 2. *Let X be a universe. Each nearness on X is $\mathcal{T}_W$ -transitive.*

Statement of the corollary is not true already for transitivity with respect to the Lukasiewicz t-norm $\mathcal{T}_L$, defined by:

$$\mathcal{T}_L(x,y) = \max(0, x+y-1) \text{ for each } x, y \in [0,1].$$

Example 4. Let X be a universe consisting of three elements: $X = \{a, b, c\}$. Let N be such a reflexive and symmetrical binary fuzzy relation on X that $N(a,c) = \frac{1}{3}, N(a,b) = N(b,c) = \frac{3}{4}$. N is a nearness, but it is not $\mathcal{T}_L$-transitive:

$$\mathcal{T}_L(N(a,b), N(b,c)) = \frac{1}{2} > \frac{1}{3} = N(a,c).$$

Problem, concerning transitivity with respect to the maximal t-norm $\mathcal{T}_M$ for shift-invariant nearnesses is solved in [3]. It is shown there, that the only T_M-transitive shift-invariant nearness is the trivial one, assuming only two values, 0 and 1.

Remark 2. It can be simply proved, that if E is a $\mathcal{T}_M$-equivalence on X and x, y, z are arbitrary points of X, then

$E(x,y) = E(y,z) = E(x,z)$, or $E(x,y) = E(y,z) < E(x,z)$, or $E(x,y) = E(x,z) < E(y,z)$, or $E(x,z) = E(y,z) < E(x,y)$.

Theorem 5. *Every $\mathcal{T}_M$-equivalence on a universe X possesses the property (N3).*

Proof. Let E be a $\mathcal{T}_M$-equivalence on X not satisfying (N3). Hence there is an $\epsilon > 0$ such that for all natural n there exist $x_n, y_n, z_n \in X$ such that $E(x_n, y_n) > 1 - \frac{1}{n}$, but $|E(x_n, z_n) - E(y_n, z_n)| \geq \epsilon$.

It follows, that for each n is $E(x_n, z_n) \neq E(y_n, z_n)$. According to Remark 2 then

$$E(x_n, y_n) = \min(E(x_n, z_n), E(y_n, z_n)) < \max(E(x_n, z_n), E(y_n, z_n)).$$

It means, that both values $E(x_n, z_n)$ and $E(y_n, z_n)$ belong to interval $(1 - \frac{1}{n}, 1]$, thus

$$|E(x_n, z_n) - E(y_n, z_n)| < \frac{1}{n}, \text{a contradiction.}$$

Corollary 3. *Let X be a universe and E be a $\mathcal{T}_M$-equivalence on X. Then E is the nearness on X.*

If X is any universe, denote by $\mathcal{A}$ the set of all $\mathcal{T}_M$-equivalences on X, by $\mathcal{B}$ the set of all nearnesses on X and by $\mathcal{C}$ the set of all $\mathcal{T}_W$-equivalences on X. We have proved:

$$\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}.$$

These inequalities are, in general, proper.

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