Fuzzy Linear Space Based on Fuzzy Subfield

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Abstract In this paper, the fuzzy linear space based on a fuzzy subfield is defined by the nested sets, it's pointwise characterization and some properties of the linear relations of fuzzy vectors are discussed.

Keyword Fuzzy linear space, Fuzzy subfield, Fuzzy linear dependent, Fuzzy base.

S. Nanda[1] defined the fuzzy fields and fuzzy linear space, R. Biswas[2] showed the mistakes in [1] and redefined the concepts of fuzzy fields and fuzzy linear space. Gu wenxiang[3] gave one example, not only showed some propositions are fale in [1], but also maintained that the definitions in [2] does not remove errors in [1], Therefore redefined the concepts. Li Kelin [4] supplemented one condition for the definition in [3] and gave some results. On this paper we discussed fuzzy linear space on fuzzy subfield by nested sets, its pointwise characterization and some properties of the linear relations of fuzzy vectors are discussed.

1. Introduction

Definition 1.1 [5] Let X be an universe and P(X) be power set of X, $\Gamma \subseteq [0, 1]$, If the mapping $H: \Gamma \to P(X)$

$$\mu \to H(\mu)$$

Satisfies: $\lambda < \mu \Leftrightarrow H(\lambda) \supseteq H(\mu)$

then H is called a nested set of X, and we denote $H_\Gamma = \{H(\lambda) \mid \lambda \in \Gamma\}$.

H_Γ can be reduced or refined according the convention in [7]. If H_Γ satisfies the condition: When $\lambda \neq \mu$ $(\lambda, \mu \in \Gamma)$, $H(\lambda) \neq H(\mu)$ (for all $\lambda, \mu \in \Gamma$), then H is called irreducible nested set.

Two nested sets in X have the relation R if and only if they have the same irreducible nested set about the same index set Γ, then the relation R is an equivalent relation.

Using relation R we can classify the all nested set in X and the class containing H_Γ is denoted $\overline{H}_\Gamma$ which is called a fuzzy subset in X, and denoted H. The set of the all fuzzy subsets on X is denoted $\mathscr{F}(X)$. We charcterized fuzzy subfield and fuzzy linear space by irreducible nested sets.

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Definition 1.2 [6] Let E be a field and $\underset{\sim}{F}\in\mathscr{F}(E)$, F_Γ be the irreducible nested sets of $\underset{\sim}{F}$. If $\forall\,\lambda\in\Gamma$, F_λ is a subfield of E, then $\underset{\sim}{F}$ is called a fuzzy subfield of E.

About the fuzzy subfield we have following conclutions: [6]

(1) If $\underset{\sim}{F}$ is a fuzzy subfield of E, then $\underset{\sim}{F}(1)=\underset{\sim}{F}(0)\geqslant\underset{\sim}{F}(x)$, $\forall\,x\in E$;

(2) $\underset{\sim}{F}$ is a fuzzy subfield of E iff $\forall\,x,y\in E$, (a) $\underset{\sim}{F}(x+y)\geqslant\underset{\sim}{F}(x)\wedge\underset{\sim}{F}(y)$, (b) $\underset{\sim}{F}(-x)\geqslant\underset{\sim}{F}(x)$, (c) $\underset{\sim}{F}(xy)\geqslant\underset{\sim}{F}(x)\wedge\underset{\sim}{F}(y)$, (d) $\underset{\sim}{F}(x^{-1}\geqslant\underset{\sim}{F}(x)$, $(x\neq 0)$;

Definition 1.3 Let $A,B\in\mathscr{F}(E)$, then the product $A\circ B$ is defined as follows:

$$A\circ B(x)=\begin{cases}\bigvee\limits_{yz=x} A(y)\wedge B(z) & \text{for } y,z\in E,\\ 0 & \text{otherwise}\end{cases}$$

An x_λ is called a fuzzy point in E iff $x_\lambda(y)=0$ for $y\neq x$ and $x_\lambda(x)=\lambda\neq 0$, $x_\lambda\circ y_\mu=(xy)_{\lambda\wedge\mu}$ is a fuzzy point too. If $A\in\mathscr{F}(E)$, $A(x)=\lambda$, $x_{A(x)}$ is called a principal element of A.

(3) $\underset{\sim}{F}$ is a fuzzy subfield iff $\forall\ x_\lambda,y_\mu\in\underset{\sim}{F}$, $x_\lambda\oplus y_\mu\in\underset{\sim}{F}$, $-(x_\lambda)\in\underset{\sim}{F}$, $x_\lambda\circ y_\mu\in\underset{\sim}{F}$, $x_\lambda^{-1}(x\neq 0)\in\underset{\sim}{F}$.

2. Fuzzy Linear Space

Definition 2.1 Let $\underset{\sim}{F}$ be a fuzzy subfield and V be a Linear space on E, $\varnothing\neq\underset{\sim}{W}\in\mathscr{F}(V)$, W_Γ be the irreducible nested sets of $\underset{\sim}{W}$. If $\forall\,\lambda\in\Gamma$, W_λ is a linear space on F_λ, then $\underset{\sim}{W}$ is called a fuzzy linear space on $\underset{\sim}{F}$. When $\underset{\sim}{F}=E$, $\underset{\sim}{W}$ is called the fuzzy linear space on E.

Example 2.2 Let $\underset{\sim}{F}\in\mathscr{F}(\mathbf{C})$ ($\mathbf{C}$ is the complex field), F_Γ is defined by

$$\Gamma:\quad 0<0.1<0.5<0.8$$

$$F_\Gamma:\quad \mathbf{C}\geqslant\mathbf{R}\geqslant\mathbf{Q}(\sqrt{2})\geqslant\mathbf{Q},$$

then $\underset{\sim}{F}$ is a fuzzy subfield of $\mathbf{C}$. As we knowen, $\mathbf{C}_n[x]=\{f(x)\mid f(x)\in\mathbf{C}[x],\deg(f(x))<n\}\cup\{0\}$ is a linear space on $\mathbf{C}$. Let $\underset{\sim}{W}\in\mathscr{F}[\mathbf{C}_n[x]]$, W_Γ is defined by

$$\Gamma:\quad 0<0.1<0.5<0.8$$

$$W_\Gamma:\mathbf{C}_n[x]\supseteq\mathbf{R}_n[x]\supseteq\mathbf{Q}(\sqrt{2})_n[x]\supseteq\mathbf{Q}_n[x],$$

then $\underset{\sim}{W}$ is a fuzzy linear space on the fuzzy subfield $\underset{\sim}{F}$ of E.

Theorem 2.3 Let V be a linear space on a field E and $\underset{\sim}{F}$ be a fuzzy subfield of E, $\underset{\sim}{W} \in \mathscr{F}(V)$. $\underset{\sim}{W}$ is a fuzzy linear space on $\underset{\sim}{F}$ iff

(1) $\forall\, \alpha, \beta \in V, \underset{\sim}{W}(\alpha-\beta) \geqslant \underset{\sim}{W}(\alpha) \wedge \underset{\sim}{W}(\beta)$;

(2) $\forall\, \alpha \in V, \forall\, k \in E, \underset{\sim}{W}(k\alpha) \geqslant \underset{\sim}{W}(\alpha) \wedge \underset{\sim}{F}(k)$.

Proof: (only proving Sufficiency) Suppose $\underset{\sim}{W} \in \mathscr{F}(V)$, W_Γ is the irreducible nested sets of $\underset{\sim}{W}$. $\forall\, \lambda \in \Gamma, \forall\, \alpha, \beta \in W_\lambda, \forall\, k \in F_\lambda$, then $\underset{\sim}{W}(x) \geqslant \lambda, \underset{\sim}{W}(\beta) \geqslant \lambda, \underset{\sim}{F}(k) \geqslant \lambda$, therefore $\underset{\sim}{W}(\alpha-\beta) \geqslant \lambda, \alpha-\beta \in W_\lambda, \underset{\sim}{W}(kx) \geqslant \lambda, kx \in W_\lambda$, W_λ is a linear space on F_λ, thus $\underset{\sim}{W}$ is a fuzzy linear space on $\underset{\sim}{F}$.

Theorem 2.4 Let V be a linear space on a field E and $\underset{\sim}{F}$ be a fuzzy subfield of E. $\underset{\sim}{W} \in \mathscr{F}(V)$ is a fuzzy linear space on $\underset{\sim}{F}$ iff $\forall\, \alpha, \beta \in V, \forall\, k, l \in E$.

$\underset{\sim}{W}(k\alpha + l\beta) \geqslant (\underset{\sim}{W}(\alpha) \wedge \underset{\sim}{W}(\beta)) \wedge (\underset{\sim}{F}(k) \wedge \underset{\sim}{F}(l))$.

Suppose that $\alpha_\lambda, \beta_\mu \in \underset{\sim}{W}, k_t, l_s \in \underset{\sim}{F}$, defining $k_t \circ \alpha_\lambda = (k\alpha)_{t\wedge\lambda}$, then We have following the pointwise characterization on fuzzy linear space.

Theorem 2.5 Let V be a linear space on a field E and $\underset{\sim}{F}$ be a fuzzy subfield of E. $\underset{\sim}{W} \in \mathscr{F}(V)$ is a fuzzy linear space on $\underset{\sim}{F}$ iff $\forall\, \alpha_\lambda, \beta_\mu \in \underset{\sim}{W}, k_t, l_s \in \underset{\sim}{F}, k_t \circ \alpha_\lambda \oplus l_s \circ \beta_\mu \in \underset{\sim}{F}$. (where "$\oplus$", "$\circ$" are the operation introduced by the operation "$+$", "$\cdot$" in E.)

Theorem 2.6

(1) If $\underset{\sim}{W}_1, \underset{\sim}{W}_2$ are fuzzy linear spaces on a fuzzy subfield $\underset{\sim}{F}$, then $\underset{\sim}{W}_1 \cap \underset{\sim}{W}_2$ so is;

(2) If $\underset{\sim}{W}_1, \underset{\sim}{W}_2$ are fuzzy linear spaces on a fuzzy subfield $\underset{\sim}{F}$, then $\underset{\sim}{W}_1 \oplus \underset{\sim}{W}_2$ so is;

(3) If $\underset{\sim}{W}$ is fuzzy linear space on $\underset{\sim}{F}$, then $\forall\, \alpha \in V, \underset{\sim}{W}(\alpha) \leqslant \underset{\sim}{W}(0) \leqslant \underset{\sim}{F}(0) = \underset{\sim}{F}(1)$;

(4) If $\underset{\sim}{W}$ is fuzzy linear space on $\underset{\sim}{F}$, $W_{\underset{\sim}{W}(0)} = \{\alpha \in V \mid \underset{\sim}{W}(\alpha) = \underset{\sim}{W}(0)\} \neq \{0\}$, then $W_{\underset{\sim}{W}(0)}$ containes the minimal nonzero subspace of V.

3. Fuzzy vector and Fuzzy linear relation

We call the elements in fuzzy linear space $\underset{\sim}{W}$ as the fuzzy vectors and the elements in fuzzy subfield $\underset{\sim}{F}$ as the fuzzy numbers. Specially the fuzzy vector O_λ is called a λ-zero vector and the fuzzy number 0_μ is called a μ-zero number.

Definition 3.1 Let $\underset{\sim}{W}$ be a fuzzy linear space on $\underset{\sim}{F}, \alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m} \in \underset{\sim}{W}$. If existing a group of fuzzy numbers $a_{\mu_1}^{(1)}, a_{\mu_2}^{(2)}, \cdots a_{\mu_m}^{(m)}$ which are not all zero in $\underset{\sim}{F}$, such that

$$a_{\mu_1}^{(1)} \alpha_{\lambda_1}^{(1)} \oplus a_{\mu_2}^{(2)} \alpha_{\lambda_2}^{(2)} \oplus \cdots \oplus a_{\mu_m}^{(m)} \alpha_{\lambda_m}^{(m)} = O_\lambda,$$

(where $\lambda = \wedge \{\lambda_i, \mu_j | i,j = 1.2, \cdots m\} > 0$) then we call fuzzy vectors $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ are λ—linear dependent ,conversely, we call they are λ—linear independent.

Note 1: $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ are λ—linear dependent means the degree of linear dependent of vectors $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(m)}$ in V is λ。

Note 2: Though fuzzy vectors $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ are λ—linear independent, but they may be μ—linear dependent ($\mu > \lambda$).

Example 3.2 Let $V = \mathbf{R}^3$ be the 3—dimension real linear space, $\underset{\sim}{F}$ be a fuzzy subfield of **R** and $\underset{\sim}{W}$ be a fuzzy linear space on $\underset{\sim}{F}$, $\underset{\sim}{F}$ and $\underset{\sim}{W}$ are defined as following:

$$\underset{\sim}{F}(x) = \begin{cases} 0.8, x \in \mathbf{Q} \\ 0.5, x \in \mathbf{Q}(\sqrt{2}) \backslash \mathbf{Q} \\ 0.1, x \in \mathbf{R} \backslash \mathbf{Q}(\sqrt{2}) \end{cases}$$

$$\underset{\sim}{W}(\alpha) = \begin{cases} 0.9, \alpha = (r_1, 0, 0) \\ 0.4, \alpha = (r_1, r_2, 0) & r_2 \neq 0 \\ 0.1, \alpha = (r_1, r_2, r_3) & r_3 \neq 0 \end{cases}$$

Let $\alpha_{\lambda_1}^{(1)} = (1,0,0)_{0.9}, \alpha_{\lambda_2}^{(2)} = (1,1,0)_{0.4}, \alpha_{\lambda_3}^{(3)} = (1,2,0)_{0.4}$, then exists fuzzy number $\sqrt{2}_{0.5}, (-2\sqrt{2})_{0.5}, (\sqrt{2})_{0.5}$, such that

$$\sqrt{2}_{0.5} \circ (1,0,0)_{0.9} \oplus (-2\sqrt{2})_{0.5} \circ (1,1,0)_{0.4} \oplus \sqrt{2}_{0.5} \circ (1,2,0)_{0.4}$$
$$= (\sqrt{2}, 0, 0)_{0.5} \oplus (-2\sqrt{2}, -2\sqrt{2}, 0)_{0.4} \oplus (\sqrt{2}, 2\sqrt{2}, 0)_{0.4}$$
$$= (0,0,0)_{0.4}$$

Thus $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \alpha_{\lambda_3}^{(3)}$ are 0.4—linear dependent. But there exist no any group of fuzzy numbers $a_{\mu_1}^{(1)}, a_{\mu_2}^{(2)}, a_{\mu_3}^{(3)}$ which are not all zero such that $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \alpha_{\lambda_3}^{(3)}$ are 0.5—linear dependent.

Note 3 When $\lambda = \underset{\sim}{W}(0)$, we call fuzzy vectors $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ are linear dependent. In this case, $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \underset{\sim}{W}(0)$。

Note 4 In a fuzzy linear space, We can define vector subtraction: $\alpha_\lambda \ominus \beta_\mu = \alpha_\lambda$

$\oplus(-\beta)_\mu$. But, it is different from classical linear space that the operation "$\oplus$" doesn't satisfy additive eliminate law. Therefore, $\alpha_\lambda \oplus \beta_\mu = r_t$ doesn't imply $\alpha_\mu = r_t \ominus \beta_\mu$.

Theorem 3.3 Let $\underset{\sim}{W}$ is a fuzzy linear space on a fuzzy subfield $\underset{\sim}{F}$ and $\alpha^{(1)}_{\lambda_1}$, $\alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}$ be fuzzy vectors in $\underset{\sim}{W}$, $\lambda = \min\{\lambda_1, \cdots, \lambda_m\}$. $\alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}$ are λ-linear dependent iff $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(m)}$ are linear dependent in W_λ.

Proof: (Neccesary) Suppose that $\alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}$ are λ-linear dependent, then exist a group of fuzzy numbers $a^{(1)}_{\mu_1}, a^{(2)}_{\mu_2}, \cdots a^{(m)}_{\mu_m}$ which are not all zero ($\mu_i \geqslant \lambda$, $i = 1, 2, \cdots m$) satisfied:

$$a^{(1)}_{\mu_1}\alpha^{(1)}_{\lambda_1} \oplus a^{(2)}_{\mu_2}\alpha^{(2)}_{\lambda_2} \oplus \cdots \oplus a^{(m)}_{\mu_m}\alpha^{(m)}_{\lambda_m} = O_\lambda,$$

Since $\lambda = \min\{\lambda_i, \mu_j \mid i, j = 1.2, \cdots m\}$, then $a^{(1)}, a^{(2)}, \cdots a^{(m)} \in F_\lambda$, $\alpha^{(1)}, \alpha^{(2)}, \cdots \alpha^{(m)} \in W_\lambda$, and in W_λ we have $a^{(1)}\alpha^{(1)} + a^{(2)}\alpha^{(2)} + \cdots + a^{(m)}\alpha^{(m)} = 0$ (*)

i.e, $\alpha^{(1)}, \alpha^{(2)}, \cdots \alpha^{(m)}$ are linear dependent in W_λ.

(Sufficiency) If $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(m)}$ are linear dependent in W_λ, the there exist a group of numbers $a^{(1)}, a^{(2)}, \cdots a^{(m)} \in F_\lambda$, such that (*) is truth, Since $\underset{\sim}{F}(a^{(i)}) \geqslant \lambda$, ($i = 1, 2, \cdots m$) So we can choose a group of fuzzy numbers $a^{(1)}_\lambda, a^{(2)}_\lambda, \cdots a^{(m)}_m \in F_\lambda$, which are not all zero, such that

$$a^{(1)}_\lambda \alpha^{(1)}_{\lambda_1} \oplus a^{(2)}_\lambda \alpha^{(2)}_{\lambda_2} \oplus \cdots a^{(m)}_\lambda \alpha^{(m)}_{\lambda_m} = O_\lambda.$$

Corollary 3.4 Let $\alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}$ be fuzzy vecters in fuzzy linear space $\underset{\sim}{W}$. If $\alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}$ are λ-linear dependent, then they must be μ-linear dependent ($\mu \leqslant \lambda$).

Corollary 3.5 If $\alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}$ are λ-linear independent, then they must be μ-linear independent ($\mu \geqslant \lambda$).

Theorem 3.6 (1) If there is μ-zero vector in the group of fuzzy vectors $\alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}$, then $\alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}$ must be are λ-linear dependent, ($\lambda = \min\{\lambda_i\}$) and there is a part of fuzzy vectors are μ-linear dependent.

(2) If $\alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}$ are μ-linear independent, the It's any non-empty part-group of fuzzy vectors are λ-linear independent.

Definition 3.7 Let $\underset{\sim}{W}$ be a fuzzy linear space on $\underset{\sim}{F}$, $\alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}, \beta_\lambda \in \underset{\sim}{W}$. If there exist $a^{(1)}_{\mu_1}, a^{(2)}_{\mu_2}, \cdots a^{(m)}_{\mu_m} \in \underset{\sim}{F}$, such that $\beta_\lambda = a^{(1)}_{\mu_1}\alpha^{(1)}_{\lambda_1} \oplus a^{(2)}_{\mu_2}\alpha^{(2)}_{\lambda_2} \oplus \cdots \oplus a^{(m)}_{\mu_m}\alpha^{(m)}_{\lambda_m}$,

we call fuzzy vector β_λ can be λ-linear represented by $\alpha^{(1)}_{\lambda_1}, \alpha^{(2)}_{\lambda_2}, \cdots \alpha^{(m)}_{\lambda_m}$ (This means the degree which β can be linear represented by $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(m)}$ is λ.)

Theorem 3.8 $\underset{\sim}{W}, \alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}, \beta_\lambda$ are above, then β_λ can be λ—linear represented by $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ iff β can be linear represented by $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(m)}$ in W_λ.

Definition 3.9 $\underset{\sim}{W}, \alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}, \beta_\lambda$ are above, if $\beta_\lambda = a_{\lambda_1}^{(1)} \alpha_{\lambda_1}^{(1)} \oplus a_{\lambda_2}^{(2)} \alpha_{\lambda_2}^{(2)} \oplus \cdots \oplus a_{\lambda_m}^{(m)} \alpha_{\lambda_m}^{(m)}$,

then we call β_λ is just λ—linear represented by $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$. In this case, β_λ is called a fuzzy linear combination of $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$.

Note 5 In classical linear space, We have a proposition:

If $\alpha_1, \alpha_2, \cdots \alpha_m$ are linear independent and $\alpha_1, \alpha_2, \cdots \alpha_m, \beta$ are linear dependent, then β must be linear represented by $\alpha_1, \alpha_2, \cdots \alpha_m$. In fuzzy linear space, this proposition is not truth.

For example: Let $V, \underset{\sim}{W}, \underset{\sim}{F}$ are above in example 3.2, Let $\alpha_{\lambda_1}^{(1)} = (1,0,0)_{0.6}, \alpha_{\lambda_2}^{(2)} = (1,1,0)_{0.5}, \alpha_{\lambda_3}^{(3)} = (1,1,1)_{0.4}, \beta_\lambda = (1,2,0)_{0.5} \in \underset{\sim}{W}$; then $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \alpha_{\lambda_3}^{(3)}$ λ—linear independent. ($\lambda \in [0,1]$) and $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \alpha_{\lambda_3}^{(3)}, \beta_\lambda$ are 0.4—linear dependent, but β_λ can not be fuzzy linear represented by $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \alpha_{\lambda_3}^{(3)}$.

Definition 3.10 Let $\underset{\sim}{W}$ be a fuzzy linear space on $\underset{\sim}{F}, \alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ and $\beta_{\mu_1}^{(1)}, \beta_{\mu_2}^{(2)}, \cdots \beta_{\mu_n}^{(n)}$ be two groups of fuzzy vectors in $\underset{\sim}{W}$. If $\forall$ i $(i=1,2,\cdots m)$, $\alpha_{\lambda_i}^{(i)}$ can be fuzzy linear represented by $\beta_{\mu_1}^{(1)}, \beta_{\mu_2}^{(2)}, \cdots \beta_{\mu_n}^{(n)}$, then we call the fuzzy vectors group of $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ can be fuzzy linear represented by the fuzzy vectors group of $\beta_{\mu_1}^{(1)}, \beta_{\mu_2}^{(2)}, \cdots \beta_{\mu_n}^{(n)}$.

If two fuzzy vectors groups can be fuzzy linear represented for each other, then we call these two fuzzy vector groups equivalent.

Theorem 3.11 The two fuzzy vector groups are above, then they are equivelent iff there exists $t \in [0,1]$, such that vector group of $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(m)}$ and vector group of $\beta^{(1)}, \beta^{(2)}, \cdots, \beta^{(m)}$ are equivalent in W_t.

Theorem 3.12 The two fuzzy vector groups are above. These two vector groups are equivalent iff (1)$\alpha^{(1)}, \cdots \alpha^{(m)}$ and $\beta^{(1)}, \cdots \beta^{(m)}$ are equivatent; (2)$\lambda_i = \mu_j = t$ for $i = 1. 2 \cdots m, j = 1. 2 \cdots n$.

Definition 3.13 Let $\underset{\sim}{W}$ be a fuzzy linear space on $\underset{\sim}{F}, \alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)} \in \underset{\sim}{W}$. If

(1)$\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_k}^{(k)} (k \leq m)$ are λ—linear independent,

(2)$\alpha_{\lambda_j}^{(j)} (j=1,2\cdots m)$ can be λ—linear represented by $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_k}^{(k)}$

Then $\{\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_k}^{(k)}\}$ is called a λ—maximal linear independent group of

$\{\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}\}$ and k is called the λ—rank of fuzzy vector group of $\{\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}\}$.

Theorem 3. 14 Let $\underset{\sim}{W}$ be a fuzzy linear space on $\underset{\sim}{F}$, $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)} \in \underset{\sim}{W}$. Then $\{\alpha_{\lambda_1}^{(1)}, \cdots \alpha_{\lambda_k}^{(k)}\}$ is a λ—maximal linear independent group of $\{\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}\}$ iff

(1)$\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(k)}\}$ is maximal linear independent group of $\{\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha^{(m)}$ in W_λ;

(2)$\lambda_j (j = k+1, \cdots m) = \min\{\lambda_1, \lambda_2, \cdots \lambda_k\}$

4. Fuzzy Bases and Fuzzy dimensions

Definition 4. 1 Let $\underset{\sim}{W}$ be a fuzzy linear space on $\underset{\sim}{F}$ and $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)} \in \underset{\sim}{W}$. If

(1)$\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ are λ—linear independent;

(2)$\forall\ \beta_\mu (\mu \geqslant \lambda) \in \underset{\sim}{W}$, β_μ can be λ—linear represented by $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$;

then $\{\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}\}$ is called a λ—fuzzy base of $\underset{\sim}{W}$ and (m, λ) is called the λ—dimension of $\underset{\sim}{W}$. Denote $\dim(\underset{\sim}{W}(\lambda)) = (m, \lambda)$.

Example 4. 2 Let $\underset{\sim}{W}, \underset{\sim}{F}$ be above in example 3. 2, then $\alpha_{\lambda_1}^{(1)} = (1, 0, 0)_{0.4}$, $\alpha_{\lambda_2}^{(2)} = (0, 1, 0)_{0.4}$ is a 0. 4—base of $\underset{\sim}{W}$. $\forall\ \beta_\mu = (a, b, o)_\mu (\mu \geqslant 0.4) \in W$,

then

$$a_\mu \alpha_{\lambda_1}^{(1)} \oplus b_\mu \alpha_{\lambda_2}^{(2)} \oplus 1_{0.4} \beta_\mu = 0_{0.4}$$

$$\dim(\underset{\sim}{W}(0.4)) = (2, 0.4)$$

By this example, We know that fuzzy linear space is not simple simalator of classical linear space, it contains richer connotation.

Theorem 4. 3 Let $\underset{\sim}{W}, \alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ be above, then $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ is a λ—base of $\underset{\sim}{W}$ iff $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(m)}$ is a base of W_λ.

Theorem 4. 4 Let $\underset{\sim}{W}$ be a fuzzy linear space on $\underset{\sim}{F}$, and $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ be a λ—base of $\underset{\sim}{W}$, then $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ can be extended as a μ—base of $\underset{\sim}{W}(\mu \leqslant \lambda)$. i. e. $\dim(\underset{\sim}{W}(\lambda)) \leqslant \dim(\underset{\sim}{W}(\mu))$ (when $\mu \leqslant \lambda$)

Theorem 4. 5 Let $\underset{\sim}{W}$ be a fuzzy linear space on $\underset{\sim}{F}$, $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ be a λ—base of $\underset{\sim}{W}$. If β_λ be just linear represented by $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$, then the represent way is unique.

Proof: Suppose that $\beta_\lambda = a_{\lambda_1}^{(1)}\ \alpha_{\lambda_1}^{(1)} \oplus\ a_{\lambda_2}^{(2)}\ \alpha_{\lambda_2}^{(2)} \oplus \cdots \oplus a_{\lambda_m}^{(m)}\ \alpha_{\lambda_m}^{(m)}$,
and $\beta_\lambda = b_{\lambda_1}^{(1)}\ \alpha_{\lambda_1}^{(1)} \oplus\ b_{\lambda_2}^{(2)}\ \alpha_{\lambda_2}^{(2)} \oplus \cdots \oplus b_{\lambda_m}^{(m)}\ \alpha_{\lambda_m}^{(m)}$,

then $(a_{\lambda_1}^{(1)} \ominus b_{\lambda_1}^{(1)})\alpha_{\lambda_m}^{(1)} \oplus \cdots \oplus (a_{\lambda_m}^{(m)} \ominus b_{\lambda_m}^{(m)})\alpha_{\lambda_m}^{(m)} = 0_\lambda$

since $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ are λ—linear independent, so $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(m)}$ are linear independent in W_λ, thus, In W_λ we have $a^{(i)} - b^{(i)} = 0$ $(i = 1, 2, \cdots m)$, i. e. $a^{(i)} = b^{(i)}, a_{\lambda_i}^{(i)} = b_{\lambda_i}^{(i)}, (i = 1, 2 \cdots m)$。

Definition 4. 6 Let $\underset{\sim}{W}$ be a fuzzy linear space on $\underset{\sim}{F}$ and $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ be a λ —base of $\underset{\sim}{W}$. If β_λ be just linear represented by $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$, i. e.

$\beta_\lambda = a_{\lambda_1}^{(1)}\ \alpha_{\lambda_1}^{(1)} \oplus a_{\lambda_2}^{(2)}\ \alpha_{\lambda_2}^{(2)} \oplus \cdots \oplus a_{\lambda_m}^{(m)}\ \alpha_{\lambda_m}^{(m)}$,

then $(a_{\lambda_1}^{(1)}, a_{\lambda_2}^{(2)}, \cdots a_{\lambda_m}^{(m)})$ is called the λ—coordinate of β_λ about $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$.

Theorem4. 7 Let $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$ be a λ—base of W, and $\lambda = \min\{\lambda_i \mid i = 1, 2 \cdots m\}$ If $a_{\lambda_1}^{(1)}, a_{\lambda_2}^{(2)}, \cdots a_{\lambda_m}^{(m)}$ is the λ—coordinate of β_λ about $\alpha_{\lambda_1}^{(1)}, \alpha_{\lambda_2}^{(2)}, \cdots \alpha_{\lambda_m}^{(m)}$, then

(1)$\forall\ \mu \leqslant \lambda, (a^{(1)}, a^{(2)}, \cdots, a^{(m)})$is the coordinate of β in W_μ;

(2)$\underset{\sim}{F}(a^i) = \underset{\sim}{W}(\alpha^{(i)})$ $(i = 1, 2 \cdots m)$.

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