ROUGH SETS ARE FUZZY SETS

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Abstract

In this paper the author says that the existing concept on rough sets to be a different topic from fuzzy sets is not correct. In fact, rough sets are fuzzy sets but the converse is not true in general.

Keywords: Fuzzy set, rough set.

1 Introduction

The long standing concern about knowledge representation to deal with vagueness or inexactness has been successfully cured with the help of a new and powerful mathematical tool designed by Zadeh [21] in this century. This is what we call the theory of 'fuzzy sets', which will surely play a vital role in the coming century. After a span of about two decades since the discovery of fuzzy set theory, another theory is modelled by Pawlak [18], which is known as theory of 'rough sets' and it has been growing as an useful tool to analize incomplete information systems. In [20], Pawlak makes a complete study of the two notions; and presents a mathematical justification to show that they are different concepts. Pawlak [20] says that the idea of rough sets can not be reduced to the idea of fuzzy sets. In another work [10], Dubois and Prade also state that fuzzy sets and rough sets are two different topics. In fact there are a number of works [viz. 2, 3, 5, 10, 14, 15, 17] combining the two notions, a proposal first suggested by Dubois and Prade [10].

In the present paper we claim that rough sets are fuzzy sets but converse is not true in general. In fact, we find an element of miscalculation in the Pawlak's work [20] which we point out here. For the preliminaries on fuzzy set theory and rough set theory, [7, 8, 9, 12, 13, 16, 17, 18, 19, 21, 22] may be seen.

2 Rough Sets Are Fuzzy Sets

Let (U, R) be an approximation space [18], and suppose that the rough set [18] of a set $X \subseteq U$ in (U, R) is A(X) given by $A(X) = (X_1, X_2)$.

For an element $u \in U$, the degree of belongingness (rough membership value) of u in X is $\mu(u)$ given by

$$\mu(u) = \frac{\#([u]_R \cap X)}{\#[u]_R}$$

where $[u]_R$ represents the equivalence class of R containing u, and the symbol # stands to represent the cardinality, as usual. Here $0 \leq \mu(u) \leq 1$.

This membership function immediately unearth the fuzzy set U_x of U given by

$$U_x = \{ (u, \mu_{U_x}(u)) : u \in U, \mu_{U_x}(u) = \frac{\#([u]_R \cap X)}{\#[u]_R} \} \quad \text{..........................(2.1)}$$

It is important to notice here that corresponding to any rough set (X_1, X_2) there exists an equivalent fuzzy set U_x which is unique; and conversely if it is known that U_x be the equivalent fuzzy set of a rough set, then the rough set (X_1, X_2) can be obtained from the fuzzy set U_x

uniquely using the formula :

$$X_1 = \{ u : u \in U, \mu_{U_x}(u) = 1 \}$$
$$X_2 = \{ u : u \in U, \mu_{U_x}(u) \neq 0 \}$$

Example 2.1

Consider the universal set U = $\{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9. \}$

Let R be an equivalence relation on U such that the family of equivalence classes is given by :

$\{ x_1, x_5 \}, \{ x_2, x_4, x_9 \}, \{ x_3, x_6 \}$ and $\{ x_7, x_8 \}$.

Take X = $\{ x_1, x_2, x_5, x_8, x_9 \}$

Then the rough set R(X) of X in the approximation space (U, R) is (X_1, X_2), where

$$X_1 = \{ x_1, x_5 \}$$
$$X_2 = \{ x_1, x_2, x_4, x_5, x_7, x_8, x_9 \}.$$

Clearly, R(x) is equivalent to the unique fuzzy set U_x of U given by

$$U_x = \left\{ \frac{x_1}{1}, \frac{x_2}{2/3}, \frac{x_3}{0}, \frac{x_4}{2/3}, \frac{x_5}{1}, \frac{x_6}{0}, \frac{x_7}{.5}, \frac{x_8}{.5}, \frac{x_9}{2/3} \right\}.$$

Conversely, if it is given that the fuzzy set

$$\mu = \left\{ \frac{x_1}{1}, \frac{x_2}{2/3}, \frac{x_3}{0}, \frac{x_4}{2/3}, \frac{x_5}{1}, \frac{x_6}{0}, \frac{x_7}{.5}, \frac{x_8}{.5}, \frac{x_9}{2/3} \right\}$$

is equivalent fuzzy set of a rough set, then this rough set can be immediately obtained which is a pair of set (A, B) given by

A = $\{ x_1, x_5 \}$

B = $\{ x_1, x_2, x_4, x_5, x_7, x_8, x_9 \}$,

and it is unique.

Conclusion 2.1

All rough sets are fuzzy sets. Besides, two different rough sets are equivalent respectively to two different fuzzy sets.

3 Fuzzy Sets Are Not Rough Sets

In the previous section we have seen that any rough set is equivalent to a unique fuzzy set, and conversely if a fuzzy set is equivalent fuzzy set of a rough set then this rough set is also unique. Question may arise whether an arbitrary fuzzy set is the equivalent fuzzy set of a rough set ? i.e., the question may arise is as below:

" Let $U_\star$ be a fuzzy set of the set U. Does there exist any equivalence relation R on U and any subset X of U such that the rough set (X_1, X_2) of X in the approximation space (U, R) is equivalent to the fuzzy set $U_\star$?" The answer is 'No'. We justify this by an example below.

Example 3.1

Consider the set U = $\{ x_1, x_2, x_3, x_4, x_5, x_6, x_7 \}$ and the fuzzy set $U_\star$ of U given by

$U_\star = \left\{ \frac{x_1}{.5}, \frac{x_2}{0}, \frac{x_3}{.5}, \frac{x_4}{1}, \frac{x_5}{.6}, \frac{x_6}{0}, \frac{x_7}{1} \right\}$.

If $U_\star$ be the equivalent fuzzy set of some rough set (X_1, X_2), then we must have

$$X_2 = \{ x_1, x_3, x_4, x_5, x_7 \}.$$

But, a careful observation reveals that whatever be the equivalent relation R on U i.e., whatever be the approximation space (U, R), there does not exist any subset X (of U) for which the upper approximation is

$$X_2 = \{ x_1, x_3, x_4, x_5, x_7 \}.$$

Because, the value $\mu_{U_\star}(x_5) = .6 = \frac{3}{5}$ (in $\frac{p}{q}$ form with positive values of p and q) implies that $\#[x_5]_R = 5$ which is impossible. It is obvious that other equivalent $\frac{p}{q}$ forms of the rational number .6 which are $\frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \cdots$, etc. need not to be accounted here due to the fact that $\#[x_5]_R \leq \#$ U.

Consequently, we draw the following conclusion.

Conclusion 3.1

A fuzzy set of a set U is not, in general, a rough set in the approximation space (U, R) for any equivalence relation R on U.

4 Final Conclusion

In this section we present three sub-sections, in the first two of which we make an analysis on the Pawlak's work [20].

4.1 On Pawlak's Work

Consider two subsets X and Y of U. Let the rough sets of X and Y in the approximation space (U, R) are respectively

A(X) = (X_1, X_2) and A(Y) = (Y_1, Y_2), where R is an equivalence relation on U.

Suppose, U_x denotes the equivalent fuzzy set of the rough set A(X) denoted by the notion

$$f [A(X)] = U_x,$$

where it is true that $\forall$ u $\in$ U,

$$\mu_{U_x} (u) = 1, \text{ if } u \in X_1$$
$$\neq 0, \text{ if if } u \in X_2$$
$$= 0, \text{ if if } u \in X_2^c$$

Thus, we have f [A(X $\cup$ Y)] = $U_{X \cup Y}$, and f [A(X $\cap$ Y)] = $U_{X \cap Y}$.

Obviously f [A(X) $\cup$ A(Y)] $\neq$ $U_{X \cup Y}$, and f [A(X) $\cap$ A(Y)] $\neq$ $U_{X \cap Y}$.

Now refer to the section.4 of [20], where Pawlak mentions the following :

"...... we shall show that such a membership function can not be extended to union and intersection of set as in the previous section (of fuzzy set theory),

....

....

$\mu_{X \cup Y}$ (x) = 1 $\Leftrightarrow$ max {μ_X (x), μ_Y (x) } =1 ....(c)

$\Leftrightarrow$ μ_X (x) =1 or μ_Y (x) =1

....

....

$\mu_{X \cap Y}$ (x) = 0 $\Leftrightarrow$ min {μ_X (x), μ_Y (x) } =0 ....(d)

$\Leftrightarrow$ μ_X (x) =0 or μ_Y (x) =0

....

....

The membership functions for the complement of sets is the same for both fuzzy sets and rough sets, ..."

We have a different observation on the above results of Pawlak.

4.2 Our Observation

The relations (c) and (d) above (used by Pawlak), if reproduced using the notions used in the present paper, will be as below (without no loss of the meanings carried by Pawlak) :

$\mu_{U_{X \cup Y}}$ (u) = 1 $\Leftrightarrow$ max { μ_{U_x} (u), μ_{U_y} (u) } = 1 ....(c1)

and

$\mu_{U_{X \cap Y}}$ (u) = 0 $\Leftrightarrow$ min { μ_{U_x} (u), μ_{U_y} (u) } = 0 ....(d1)

A careful observation unearths that (c1) and (d1) are not correct. Because the union/intersection operation on the left-hand-side of (c1)/(d1) are not fuzzy union/intersection. Besides f [A(X) ∪ A(Y)] $\neq U_{X \cup Y}$ and f [A(X) ∩ A(Y)] $\neq U_{X \cap Y}$. In fact, $U_X \cup U_Y$ (here it is a fuzzy union) may not be equivalent fuzzy set of any rough set.

Thus there is no justification to disagree that the membership function as used in (2.1) is of Zadeh's type.

4.3 Then, Why are Rough Sets?

Rough sets are fuzzy sets, although the converse is not true in general. In fact, rough sets are very useful alltropes, of fuzzy sets, like diamonds, graphites are nice allotropes of carbons. Huge application-potential have been found to exist in-built in the design of rough sets as the amount of literatures reported on rough systems (few of which are available in [17, 19]) reveal.

It may not be irrelevant to mention, in addition, that intuitionistic fuzzy sets of Atanassov [1] have useful allotropes like i-v fuzzy sets (interval-valued fuzzy sets) as justified in [4]. But the recently developed vague sets [11] are exactly identical with the intuitionistic fuzzy sets (as justified in [6]) which follows directly from their definitions.

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