On fuzzy Abelian subgroups

Yunjie ZHAND and Dong YU
Depatrment of Basic Science, Dalian Maritime University
Dalian, Liaoning 116026, People's Republic of China

Abstract: In [6] Bhattacharya and Mukherjee introduced the concept of fuzzy Abelian subgroups. In this paper, the investigation of fuzzy Abelian subgroups is continued.
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1. Preliminaries

We first recall some basic definitions for the sake of completeness.

Definition 1.1 Let G be a group. A fuzzy subset μ of G is called a fuzzy subgroup of G if

(1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, $\forall\, x, y \in G$;

(2) $\mu(x^{-1}) \geq \mu(x)$, $\forall\, x \in G$.

The identity element of any group will always be denoted by e. If μ is a fuzzy subgroup of a group G, it follows that

$\mu(e) \geq \mu(x)$, $\forall\, x \in G$,

$\mu(x^{-1}) = \mu(x)$, $\forall\, x \in G$,

and μ_λ is called a level subgroup of μ, where μ_λ is a level subset of μ, defined by

$$\mu_\lambda = \{\, x \in G \mid \mu(x) \geq \lambda,\ \lambda \in [0, 1]\}.$$

Definition 1.2 [2] Let G be a group. A fuzzy subgroup μ of G is called to be normal if

$$\mu(xyx^{-1}) \geq \mu(y),\ \forall\, x, y \in G.$$

Theorem 1.1 [2] Let μ be a fuzzy subgroup of a group G. Then μ is a normal fuzzy subgroup of if and only if any one of the following conditions is satisfied

(1) $\mu(xyx^{-1}) = \mu(y)$, $\forall\, x, y \in G$;

(2) $\mu(xy) = \mu(yx)$, $\forall\, x, y \in G$.

Theorem 1.2 [2, 3] Let G be a group and μ a fuzzy subset of G. Then μ is a (normal) fuzzy subgroup of G is and only if the level subsets μ_λ, for $\lambda \in [0, 1]$, are (normal) subgroup of G.

Definition 1.3 [2] Let G be a group and μ a fuzzy subgroup of G. Let

$$N(\mu) = \{a \in G \mid \mu(aya^{-1}) = \mu(x),\ \forall\, x \in G\}.$$

Then $N(\mu)$ is called the fuzzy normalizer of μ.

Theorem 1.3 [2, 5] Let μ be a fuzzy subgroup of a group G. Then

(1) $N(\mu)$ is a subgroup of G;

(2) μ is fuzzy normal $\Leftrightarrow N(\mu) = G$;

(3) μ is a fuzzy normal subgroup of the group $N(\mu)$.

Definition 1.4 [7] Let μ be a fuzzy subgroup of a group G. Let

$$C(\mu) = \{a \in G \mid \mu([a, x]) = \mu(e),\ \forall\, x \in G\}.$$

Then $C(\mu)$ is called the fuzzy centralizer of μ. Where $[x, y]$ is the commutator of two element x, y in

G, that is, $[x, y] = x^{-1}y^{-1}xy$.

Theorem 1.4 [7] Let μ be a fuzzy subgroup of a group G. Then

(1) $C(\mu)$ is a subgroup of G;

(2) $C(\mu)$ is normal subgroup of $N(\mu)$.

Definition 1.5 [8] For each $i = 1, 2, \ldots, n$, let μ_i be a fuzzy subgroup of the group G_i . The product of μ_i ($i = 1, 2, \ldots, n$) is the function

$$\mu_1 \times \mu_2 \times \ldots \times \mu_n: G_1 \times G_2 \times \ldots \times G_n \rightarrow [0, 1]$$

defined by

$$\mu_1 \times \mu_2 \times \ldots \times \mu_n(x) = \min\{\mu_1(x), \mu_2(x), \ldots, \mu_n(x)\},$$

where $x = (x_1, x_2, \ldots, x_n)$, $x_i \in G$, $i = 1, 2, \ldots, n$.

Theorem 1.5 [8] For each $i = 1, 2, \ldots, n$, let μ_i be a fuzzy subgroup of the group G_i. The $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ is a fuzzy subgroup of the group $G_1 \times G_2 \times \ldots \times G_n$.

Definition 1.6 [6] Let μ be a fuzzy subgroups of a group G. Let $H = \{x \in G \mid \mu(x) = \mu(e)\}$. Then μ is fuzzy Abelian if H is an Abelian subgroup of G.

2. Main results

From here on we make $H = \{x \in G \mid \mu(x) = \mu(e)\}$.

Proposition 2.1 Let μ be a normal fuzzy subgroup of a group G. Then $H \subseteq C(\mu)$.

Proof. Since $\forall\, x \in H$ we have $\mu(x) = \mu(e)$, $\forall\, y \in G$,

$$\begin{aligned}\mu([x, y]) &= \mu(x^{-1}y^{-1}xy) \geq \min\{\mu(x), \mu(y^{-1}xy)\} \\ &= \min\{\mu(x), \mu(x)\} \text{ (Theorem 1.1)} \\ &= \mu(x) = \mu(e).\end{aligned}$$

This implies that $x \in C(\mu)$, that is, $H \subseteq C(\mu)$.

Proposition 2.2 Let μ be a fuzzy subgroup of a group G and n a natural number. Then $\mu((xy)^n) = \mu(x^n y^n)$, $\forall\, x, y \in C(\mu)$.

Proof. From Theorem 1.3(3) and Theorem 1.4(2), μ is a normal fuzzy subgroup of the group $C(\mu)$. Consequently, for $\forall\, x, y \in C(\mu)$, we have

$$\begin{aligned}\mu((xy)^n) &= \mu(xy\ldots xyxyxy) = \mu(xy\ldots xyxy^2x[x, y]) \\ &\geq \min\{\mu(xy\ldots xyxy^2x), \mu([x, y])\} \\ &= \mu(xy\ldots xyxy^2x) = \mu(x^2y\ldots xyxy^2) \\ &= \mu(x^2y\ldots xy^3x[x, y]) \geq \mu(x^3y\ldots xy^3) \\ &\geq \ldots \geq \mu(x^{n-1}yxy^{n-1}) \\ &= \mu(x^{n-1}y^n x[x, y^{n-1}]) \\ &\geq \mu(x^{n-1}y^n x) = \mu(x^n y^n)\end{aligned}$$

and

$$\begin{aligned}\mu(x^n y^n) &= \mu(x^{n-1}y^n x) = \mu(x^{n-1}yxy^{n-1}[y^{n-1}, x]) \\ &\geq \mu(x^{n-1}yxy^{n-1}) \\ &\geq \ldots \geq \mu(xy\ldots xyxy^2x) \\ &= \mu(xy\ldots xyxyxy[x, y]) \geq \mu((xy)^n).\end{aligned}$$

This is, $\mu((xy)^n) = \mu(x^n y^n)$.

Theorem 2.1 For each $i = 1, 2, \ldots, n$, let μ_i be a fuzzy Abelian group of the group G_i and $\mu_i(e_i) = \mu_j(e_j)$, $\forall i, j \in \{1, 2, \ldots, n\}$. Then $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ is a fuzzy Abelian subgroup of $G_1 \times G_2 \times \ldots \times G_n$.

Where e_i is the identity element of G_i.

Proof. From Theorem 1.5, $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ is fuzzy subgroup of $G_1 \times G_2 \times \ldots \times G_n$. For each $i = 1, 2, \ldots, n$, let

$$H_i = \{x \in G_i \mid \mu_i(x) = \mu_i(e_i)\}.$$

Then H_i is a Abelian subgroup of G_i and $H_1 \times H_2 \times \ldots \times H_n$ is a subgroup of $G_1 \times G_2 \times \ldots \times G_n$. Let

$$H = \{x \in G_1 \times G_2 \times \ldots \times G_n \mid \mu_1 \times \mu_2 \times \ldots \times \mu_n(x) = \mu_1 \times \mu_2 \times \ldots \times \mu_n(e)\},$$

$e = (e_1, e_2, \ldots, e_n)$ is the identity element of $G_1 \times G_2 \times \ldots \times G_n$. Then H is a subgroup of $G_1 \times G_2 \times \ldots \times G_n$.

Now $\forall\, x \in H_1 \times H_2 \times \ldots \times H_n$, $x = (x_1, x_2, \ldots, x_n)$, $x_i \in H_i$, we have

$$\begin{aligned}\mu_1 \times \mu_2 \times \ldots \times \mu_n(x) &= \mu_1 \times \mu_2 \times \ldots \times \mu_n(x_1, x_2, \ldots, x_n)\\ &= \min\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)\}\\ &= \min\{\mu_1(e_1), \mu_2(e_2), \ldots, \mu_n(e_n)\}\\ &= \mu_1 \times \mu_2 \times \ldots \times \mu_n(e_1, e_2, \ldots, e_n)\\ &= \mu_1 \times \mu_2 \times \ldots \times \mu_n(e).\end{aligned}$$

Consequently, $x \in H$. Conversely, $\forall\, x \in H$, $x = (x_1, x_2, \ldots, x_n)$, $x_i \in H_i$, we have

$$\begin{aligned}\min\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)\} &= \mu_1 \times \mu_2 \times \ldots \times \mu_n(x)\\ &= \mu_1 \times \mu_2 \times \ldots \times \mu_n(e)\\ &= \min\{\mu_1(e_1), \mu_2(e_2), \ldots, \mu_n(e_n)\}\\ &= \mu_i(e_i).\end{aligned}$$

It follows that $x_i \in H_i$ $(i = 1, 2, \ldots, n)$, that is, $x \in H_1 \times H_2 \times \ldots \times H_n$. Thus $H = H_1 \times H_2 \times \ldots \times H_n$.

Also, H_i is Abelian for each $i = 1, 2, \ldots, n$, so we have that $H = H_1 \times H_2 \times \ldots \times H_n$ is Abelian implying that $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ is a fuzzy Abelian subgroup of $G_1 \times G_2 \times \ldots \times G_n$ is a fuzzy Abelian subgroup of Definition 1.6.

Theorem 2.2 Let μ_α, for $\forall \alpha \in \Gamma$, be fuzzy Abelian subgroup of a group G and $\mu_\alpha(e) = \mu_\beta(e)$ for $\forall \alpha, \beta \in \Gamma$. Then $\bigcap_{\alpha \in \Gamma} \mu_\alpha$ is also fuzzy Abelian subgroup of G.

Proof. Let

$$H_\alpha = \{x \in G \mid \mu_\alpha(x) = \mu_\alpha(e)\},\ \alpha \in \Gamma,$$

$$H = \{x \in G \mid (\bigcap_{\alpha \in \Gamma} \mu_\alpha)(x) = (\bigcap_{\alpha \in \Gamma} \mu_\alpha)(e)\}.$$

Then $\forall\, x \in H$, we have

$$\mu_{\alpha'}(x) \geq \inf_{\alpha \in \Gamma} \mu_\alpha(x) = (\bigcap_{\alpha \in \Gamma} \mu_\alpha)(x) = (\bigcap_{\alpha \in \Gamma} \mu_\alpha)(e)$$

$$= \inf_{\alpha \in \Gamma} \mu_\alpha(e) = \mu_{\alpha'}(e),\ \forall \alpha' \in \Gamma.$$

That is, $x \in H_\alpha$; Conservely, $\forall\, x \in H_\alpha$, we have

$$\mu_\alpha(x) = \mu_\alpha(e)\},\ \forall \alpha \in \Gamma$$

implying that

$$(\bigcap_{\alpha \in \Gamma} \mu_\alpha)(x) = \inf_{\alpha \in \Gamma} \mu_\alpha(x) = \inf_{\alpha \in \Gamma} \mu_\alpha(e) = (\bigcap_{\alpha \in \Gamma} \mu_\alpha)(e).$$

That is, $x \in H$. It follows that $H = H_\alpha$.

Since H_α, for $\forall \alpha \in \Gamma$, is Abelian by Definition 1.6, we have that H is Abelian implying that $\bigcap_{\alpha \in \Gamma} \mu_\alpha$ is fuzzy Abelian.

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