Some Results On BZMV^{dM} - algebras

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Abstract: A BZMV^{dM} - alegbra is a system endowed with a commutative and associative binary operator $\oplus$ and two unusualorthocomplementation: a kleen orthocomplementation $\neg$ and a Brouwerian one $\sim$, and every BZMV^{dM} - algebra is both an MV- algebra and a distributive de-Morgan BZ-lattice. Our main results is that the following properties hold, for any BZMV^{dM} - algebra $(A,\oplus,\neg,\sim,0), \forall x,y\in A$, (1) $\sim\sim x = x$ iff $\sim x\otimes x = x$ iff $x\oplus x= x$, (2) $x\wedge y$=0 iff $x\leq \neg y$, (3) $\sim(x\oplus y) = \sim x \otimes \sim y$, (4) $x\oplus \sim x = 1$.
Keywords: BZMV^{dM} -algebra;Kleen orthocomplementation; Brouwerian orthocomplementation.

In 1997,G.cattaneo, R. giuntini and R. Pilla presented the concept of BZMV^{dM} - algebras[1]. The main properties of this system is studied, from [1], any BZMV^{dM} - algebras are the same as the Stonian MV– algebras and a first representation theorem is proved.

The aim of this paper is to make further investigation of the features of BZMV^{dM} - algrbras.

1. Preliminaries

Definition 1.1[1] A BZMV^{dM} - algebra is a system $<A,\oplus,\neg,\sim,0>$ where A is a non-empty set of elements, 0 is a constant element of A, $\neg$ and $\sim$ are unary operations on A, $\oplus$ is a binary operation on A, obeying the following axioms:

(MB$_1$) $(x\oplus y)\oplus z = (y\oplus z)\oplus x$,

(MB$_2$) $x\oplus 0 = x$,

(MB$_3$) $\neg(\neg x) = x$,

(MB$_4$) $\neg(\neg x\oplus y)\oplus y = \neg(x\oplus\neg y)\oplus x$,

(MB$_5$) $\sim x\oplus \sim\sim x = \sim\sim x$,

(MB$_6$) $x\oplus\sim\sim x = \sim\sim x$,

(MB$_7$) $\sim\neg[\neg(x\oplus\neg y)\oplus\neg y] = \neg(\sim\sim x\oplus\neg\sim\sim y)\oplus\neg\sim\sim y$.

From [1], making use of BZMV^{dM} - algebra structure, we can define some new operations:

$$1 := \neg 0, \qquad (1)$$

$$x \otimes y := \neg(\neg x \oplus \neg y), \qquad (2)$$

$$x \vee y := (x \otimes \neg y) \oplus y = \neg(\neg x \oplus y) \oplus y, \qquad (3)$$

$$x \wedge y := (x \oplus \neg y) \otimes y = \neg[\neg(x \oplus \neg y) \oplus \neg y]. \qquad (4)$$

Theorem1.2[1] If A is a BZMV^{dM} - algebrá, then the following results are true:

(5) $x \oplus y = y \oplus x$,

(6) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,

(7) $x \oplus 1 = 1$,

(8) $x \oplus \neg x = 1$,

(9) $\neg(x \oplus \sim\sim x) \oplus \sim\sim x = 1$,

(10) $\neg x \oplus \sim\sim x = 1$,

(11) $x \wedge \sim\sim x = x$,

(12) $\sim(x \wedge y) = \sim x \vee \sim y$,

(13) $\sim(x \vee y) = \sim x \wedge \sim y, (i.e., x \leq y \Rightarrow \sim y \leq \sim x,)$

(14) $x \wedge \sim x = 0$,

(15) $\sim\sim\sim x = \sim x$,

(16) $\sim x \oplus \sim x = \sim x$,

(17) $\neg 0 = \sim 0$.

2. Some Results

Theorem 2.1 Let A be a BZMV^{dM} -algebra, then we have

(18) $\forall x, y \in A, x \wedge y = 0$ iff $y < \sim x$

(19) for x ,and $x \oplus x = x$,then $\forall\ y \in A, x \wedge y = 0$ iff $x \leq \neg\ y$

Proof. (18) Assume $x \wedge y = 0$, then $y \wedge \sim x = (y \wedge \sim x) \vee 0 = (y \wedge \sim x) \vee (y \wedge \sim y)$ $= (\sim(x \wedge y)) = y \wedge \sim 0 = y$.

Conversely, suppose $y \leq \sim x$, then trivially

$x \wedge y = x \wedge (y \wedge \sim x) = y \wedge (x \wedge \sim x) = y \wedge 0 = 0.$

(19) Suppose $x=x\oplus x$, and, by (1) of proposition 2.2 of [1], $x\otimes y=x\wedge y$, hence, $x\leq\neg y$ iff $x\wedge y=0$.

Theorem 2.2 Let A be a BZMV^{dM} -algebra, then $\forall x,y\in A$,we have

$$\sim(x\oplus y)=\sim x\otimes\sim y$$

Proof. By theorem 1.2,we have $(x\oplus y)\wedge\sim(x\oplus y)=0$, and $\sim(x\oplus y)\leq\sim x\otimes\sim y$.

By $x\oplus y=\sim\sim x\oplus\sim\sim y$, hence $\sim(\sim\sim x\oplus\sim\sim y)=\sim(x\oplus y)$. That is

$\sim(\sim\sim x\oplus\sim\sim y)=(\sim\sim x\oplus\sim\sim y)=\neg(\neg\sim x\oplus\neg\sim y)=\sim x\otimes\sim y$.

Theorem 2.3 Let A be a BZMV^{dM} -algebra, then $\forall x\in A$, we have

$$x\oplus\sim x=1$$

Proof. Since $\sim(x\oplus\sim x)=\sim x\otimes\sim\sim x=\neg(\neg\sim x\oplus\neg\sim\sim x)=\neg(\neg\sim x\oplus\neg\neg\sim x)$.

By (8) of theorem 1.2,we have $\neg(\neg\sim x\oplus\neg\neg\sim x)=\neg 1=0$.So, $x\oplus\sim x=1$.

Definition 2.4 Let A be a BZMV^{dM} -algebra, then $\forall x,y\in A$, we introduce the following new operation "$\to$":

$$x\to y=\neg x\oplus y$$

Theorem 2.5 Let A be a BZMV^{dM} -algebra, then $\forall x,y,z\in A$, the following holds:

(20) $x\to(y\to z)=y\to(x\to z)$,

(21) $x\to y=\neg y\to\neg x$,

(22) $(x\to y)\to y=(y\to x)\to x$,

(23) $x\to x=1$.

Proof. (20) $x\to(y\to z)=\neg x\oplus(y\to z)=\neg x\oplus(\neg y\oplus z)=(\neg x\oplus\neg y)\oplus z$

$=(\neg y\oplus\neg x)\oplus z=\neg y\oplus(\neg x\oplus z)=\neg y\oplus(x\to z)$

$=y\to(x\to z)$.

(21) $x\to y=\neg x\oplus y=y\oplus\neg x=\neg(\neg y)\oplus\neg x=\neg y\to\neg x$.

(22) $(x\to y)\to y=\neg(x\to y)\oplus y=\neg(\neg x\oplus y)\oplus y=\neg(x\oplus\neg y)\oplus x$

$=\neg(\neg y\oplus x)\oplus x=(\neg y\oplus x)\to x=(y\to x)\to x$.

(23) $x\to x=\neg x\oplus x=1$

In any BZMV^{dM} - algebra the elements which are idempotent with respect to the operation $\oplus$, the condition leads to the following:

Theorem 2.6 Let A be a BZMV^{dM} -algebra, then $\forall x, y, z \in A$, the following holds:

(24) $x \to (y \oplus z) = (x \to y) \oplus (x \to z)$,

(25) $\neg x \to x = x$.

Proof. (24) Since $(x \to y) \oplus (x \to z) = (\neg x \oplus y) \oplus (\neg x \oplus z)$

$$=(\neg x \oplus \neg x) \oplus (y \oplus z) = \neg x \oplus (y \oplus z) = x \to (y \oplus z).$$

(25) $\neg x \to x = \neg \neg x \oplus x = x \oplus x = x$.

Let us introduce the following order relation "$\leq$":

$$x \leq y \text{ iff } x \to y = 1 \text{ iff } \neg x \oplus y = 1$$

Theorem 2.7 Let A be a BZMV^{dM} -algebra, then $\forall x, y, z \in A$, the following holds:

(i) if $x \leq y$, then $y \to z \leq x \to z$,

(ii) if $x \leq y$, then $\neg y \oplus z \leq \neg x \oplus z$.

Proof. (i) Since $(y \to z) \to (x \to z) = x \to ((y \to z) \to z) = x \to ((z \to y) \to y)$

$$=(z \to y) \to (x \to y) = (z \to y) \to 1 = \neg (z \to y) \oplus \neg 0 = \neg 0 = 1,$$

so, $y \to z \leq x \to z$.

(ii) Since $(\neg y \oplus z) \to (\neg x \oplus z) = (y \to z) \to (x \to z) = x \to ((y \to z) \to z) = 1$,

hence $\neg y \oplus z \leq \neg x \oplus z$.

References

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