Grey Subring and Relation of Grey Homomorphism rings *

Qiao Bingwu
(Department of Computer, Liaocheng Teacher's university,Liaocheng,252059)

Abstract: IN this paper,we introduce the concepts of grey subring and give the important properites of it.

Keywords: Grey subset, Grey Subring, Homorphism.

1 GREY SUBRING AND ITS OPERATIONS

Let R be any set and L a bounded lattice with 1 and 0,then a grey set A in R is characterized by two mapping: $\overline{U}_A : R \to L, \underline{U}_A : R \to L$ where $\overline{U}_A \geq \underline{U}_A$ .The set

$$A_{[\lambda 1, \lambda 2]} = \{x | x \in R,\ \overline{U}_A(x) \geq \lambda_2, \underline{U}_A \geq \lambda_1\}$$

is called a $[\lambda_1, \lambda_2]$—level subset of grey subset A.

Unless specially stated, R in this article only refers to the ring.

Definition 1.1 The grey subset A of R is called a grey subring, if for all $x, y \in R$, we have

(1) $\overline{U}_A(x+y) \geq \min\{\overline{U}_A(x),\ \overline{U}_A(y)\}$, $\underline{U}_A(x+y) \geq \min\{\underline{U}_A(x),\ \underline{U}_A(y)\}$;

(2) $\overline{U}_A(x \cdot y) \geq \min\{\overline{U}_A(x),\ \overline{U}_A(y)\}$, $\underline{U}_A(x \cdot y) \geq \min\{\underline{U}_A(x),\ \underline{U}_A(y)\}$;

(3) $\overline{U}_A(x) = \overline{U}_A(-x)$, $\underline{U}_A(x) = \underline{U}_A(-x)$;

(4) $\overline{U}_A(0) = \underline{U}_A(0) = 1$.

Theorem 1.2 Let A be a grey subset of R , then A is a grey subring of R iff $\forall\ [\lambda_1, \lambda_2] \subseteq L$, if $A_{[\lambda_1, \lambda_2]} \neq \varnothing$, $A_{[\lambda_1, \lambda_2]}$ is a grey subring of R.

Definition 1.3 Let R be a ring, A and B be grey subsets of R, define grey sebset of R: $A \cap B$, $A+B$, $A \cdot B$, $-A$ by:

(1) $\overline{U}_{A \cap B}(x) = \min\{\overline{U}_A(x), \overline{U}_B(x)\}$, $\underline{U}_{A \cap B}(x) = \min\{\underline{U}_A(x), \underline{U}_B(x)\}$;

(2) $\overline{U}_{A+B}(x) = \sup\limits_{x1+x2=x} \{\min\{\overline{U}_A(x_1), \overline{U}_B(x_2)\}\}$,

$\underline{U}_{A+B}(x) = \sup\limits_{x1+x2=x} \{\min\{\underline{U}_A(x_1), \underline{U}_B(x_2)\}$;

(3) $\overline{U}_{A \bullet B}(x) = \sup\limits_{x1 \bullet x2=x} \{\min\{\overline{U}_A(x_1), \overline{U}_B(x_2)\}\}$,

$\underline{U}_{A \bullet B}(x) = \sup\limits_{x1 \bullet x2=x} \{\min\{\underline{U}_A(x_1), \underline{U}_B(x_2)\}\}$;

(4) $\overline{U}_{-A}(x) = \overline{U}_A(-x)$, $\underline{U}_{-A}(x) = \underline{U}_A(-x)$;

respectively.

Theorem 1.4 Let L be a complete lattice, then $\bigcap\limits_{k \in I} A^{(k)}$ is a grey subring of R, here $\{A^{(k)}\}_{k \in I}$ is a family of grey subrings of R. If I is a finite set,then for all $[\lambda_1, \lambda_2] \subseteq L$, $(\bigcap\limits_{k \in I} A^{(k)})_{[\lambda 1, \lambda 2]} = \bigcap\limits_{k \in I} A^{(k)}_{[\lambda_1, \lambda_2]}$.

The oreml 1.5 The grey subset A of ring R is a grey subring of R if

(1) $\sup\{\overline{U}_A(x) | x \in R\} = 1$, $\sup\{\underline{U}_A(x) | x \in R\} = 1$;

* It is imbursed by Shan Dong Natural Science Fund

(2) $A+A\subseteq A$, that is, $\overline{U}_{A+A}\subseteq\overline{U}_A, \underline{U}_{A+A}\subseteq\underline{U}_A$;

(3) $-A\subseteq A$, that is, $\overline{U}_{-A}\subseteq\overline{U}_A, \underline{U}_{-A}\subseteq\underline{U}_A$;

(4) $A\cdot A\subseteq A$, that is, $\overline{U}_{A\bullet A}\subseteq\overline{U}_A, \underline{U}_{A\bullet A}\subseteq\underline{U}_A$;

Proof: If A is a grey subring of R , then $\overline{U}_A(0)=1$, $\underline{U}_A(0)=1$, thus sup $\{\overline{U}_A(x)|x\in R\}=1$, sup $\{\underline{U}_A(x)|x\in R\}=1$.

Furthermore, for any $x\in R$,

$$\overline{U}_{A+A}(x)=\sup\{\min\{\overline{U}_A(x_1), \overline{U}_A(x_2)|x=x_1+x_2\}$$
$$\leq \sup\{\overline{U}_A(x_1+x_2)| x_1+x_2=x\}=\overline{U}_A(x),$$

so, $\overline{U}_{A+A}\subseteq\overline{U}_A$. Similarly $\underline{U}_{A+A}\subseteq\underline{U}_A$.

That is $A+A\subseteq A$.

Similarly $-A\subseteq A, A\cdot A\subseteq A$.

On the contrary, conditions(1),(2),(3)and (4)hold,we prove A to be a grey subring of R, because $A+A\subseteq A, -A\subseteq A, A\cdot A\subseteq A$, so we have:

$$\overline{U}_A(x+y)\geq\overline{U}_{A+A}(x+y)$$
$$=\sup\{\inf\{\overline{U}_A(x_1),\overline{U}_A(y_1)\}| x_1+y_1=x+y\}$$
$$\geq\min\{\overline{U}_A(x),\overline{U}_A(y)\},$$
$$\overline{U}_A(x\cdot y)=\overline{U}_{A\bullet A}(x+y)$$
$$=\sup\{\inf\{\overline{U}_A(x_1),\overline{U}_A(y_1)\}| x_1\cdot y_1=x\cdot y\}$$
$$\geq\min\{\overline{U}_A(x),\overline{U}_A(y)\},$$
$$\overline{U}_A(-x)\geq\overline{U}_{-A}(-x)= U_A(x)$$

for all $x,y\in R$.

Similarly,we have

$$\underline{U}_A(x+y)\geq\min\{\underline{U}_A(x), \underline{U}_A(y)\};$$
$$\underline{U}_A(x\cdot y)\geq\min\{\underline{U}_A(x), \underline{U}_A(y)\};$$
$$\underline{U}_A(x)\geq\underline{U}_A(-x)=U_A(x);$$

for all $x,y\in R$.

Since for all $x\in R$, $0= x+(-x)$, consequently, $\overline{U}_A(0)=\overline{U}_A(x+(-x))\geq\min\{U_A(x),\overline{U}_A(-x)\}\geq\overline{U}_A(x)$, so $\overline{U}_A(0)=1$. Similarly, $\underline{U}_A(0)=1$. Hence , A is a grey subring.

Theorem1.6 Let L be a complete distributive lattice, A, B and C be grey subsets of R,then

(1) $(A+B)+C=A+(B+C)$,

(2) $A\cdot(B+C)=A\cdot B+A\cdot C$,

(3) $-(A+B)=-A+(-B)$.

Theorem1.7 Let A and B be grey subrings of R, then $A+B$ is a grey subring of R.

Theorem1.6 and Theorem1.7 can be easily drawn.

2 RELATION OF GREY HOMOMORPHISM RINGS

Let R and R' be two rings, f is a homomorphic mapping from R to R', $C_G(R)$ stands for the set composed of all the grey subrings of R. $C_G(R')$ stands for the subrings of R'.Let $A\in C_G(R)$,then $f(A)$ is defined by:

$$\overline{U}_{f(A)}(x')=\begin{cases}\sup\{\overline{U}_A(x)\mid x\in f^{-1}(x'), f^{-1}(x')\neq\phi,\\ 0 \quad if \quad f^{-1}(x')=\phi,\end{cases}$$

$$\underline{U}_{f(A)}(x')=\begin{cases}\sup\{\underline{U}_A(x)\mid x\in f^{-1}(x'), f^{-1}(x')\neq\phi,\\ 0 \quad if \quad f^{-1}(x')=\phi,\end{cases}$$

for all $x'\in R'$.Let $A'\in C_G(R')$, then $f^{-1}(A')$ is defined by:

$$\overline{U}_{f^{-1}(A')}(x)=\overline{U}_{A'}(f(x)),$$

$$\underline{U}_{f^{-1}(A')}(x)=\underline{U}_{A'}(f(x)),$$

for all $x\in R$.

Proposition2.1 Let $A, A_1, A_2 \in C_G(R), A', A'_1, A'_2\in C_G(R')$, then

(1) $\overline{U}_{f(A)}(0')=\underline{U}_{f(A)}(0')=1$,

(2) $f(f^{-1}(A')=A'$,

(3) If $A_1\subseteq A_2$,then $f(A_1)\subseteq f(A_2)$,

(4) If $A'_1\subseteq A'_2$, then $f^{-1}(A'_1)\subseteq f^{-1}(A'_2)$,

(5) $f(A_1\cap A_2)=f(A_1)\cap f(A_2)$,

(6) If $\overline{U}_{A1}(x)=1$, $\underline{U}_{A1}(x)=1, \overline{U}_{A2}(x)=1$, $\underline{U}_{A2}(x)=1$, for all $x\in$her f,
then $f(A_1+A_2)=f(A_1)+f(A_2)$,

(7) If $A'=f(A)$ and $\overline{U}_A(x)=\underline{U}_A(x)=1$ for all $x\in$her f, then $A=f^{-1}(A')$,

(8) Let $A_0=\{x\mid x\in A, \overline{U}_A(x)=\underline{U}_A(x)=1\}$,then $f(A_0)\subseteq(f(A))_0$,

(9) If $\overline{U}_A(x)=\underline{U}_A(x)=1$,for all $x\in$her f,
then $\overline{U}_{f(A)}(f(x))=\overline{U}_A(x)$, $\underline{U}_{f(A)}(f(x))=\underline{U}_A(x)$.

Proof: (1) Because $0\in\overline{U}_{f(A)}(0')$ and for any $x\in R$, $\overline{U}_A(0)=1\geq\overline{U}_A(x)$, then $\overline{U}_{f(A)}(0')=\sup\{\overline{U}_A(x)|x\in\overline{U}_{f(A)}(0')\}=\overline{U}_A(0)=1$. Similarly we have $\underline{U}_{f(A)}(0')=1$.Then $\overline{U}_{f(A)}(0')=\underline{U}_{f(A)}(0')=1$.

(2)—(8) can be easily established by[2].

(9) If $f(x)=x'$, then $\overline{U}_{f(A)}(x')=\sup\{\overline{U}_A(y)|\ y\in f^{-1}(x')\}$, and we have $\underline{U}_{f(A)}(x')=\sup\{\underline{U}_A(y)|y\in f^{-1}(x')\}$, for all $z\in f^{-1}(x')$, z-$x\in$her f and $\overline{U}_{f(A)}(z\text{-}x)=\underline{U}_{f(A)}(z\text{-}x)=1$, thus $\overline{U}_A(z\text{-}x)=1$ and $\overline{U}_A(x)=\overline{U}_A(z)$, $\underline{U}_A(x)=\underline{U}_A(z)$.

Theorem 2.2 If $A\in C_G(R), A'\in C_G(R')$, then

(1) $f^{-1}(A')$ is a grey subring of R ,and $\overline{U}_{f^{-1}(A)}(x)=\underline{U}_{f^{-1}(A)}(x)=1$, for all $x\in$her f,

(2) $f^{-1}(A'_{[0,0]})=f^{-1}(A')_{[0,0]}$,

(3) If A is constant 1 on her f,then $f^{-1}(f(A))=A$.

Proof: (1) Because A' is a grey subring of R',so $\overline{U}_{f^{-1}(A')}(x\text{-}y)=\overline{U}_{A'}(f(x\text{-}y))$ $=\overline{U}_{A'}(f(x)\text{-}f(y))\geq\overline{U}_{A'}(f(x))\wedge\overline{U}_{A'}(f(y))=\overline{U}_{f^{-1}(A')}(x)\wedge\overline{U}_{f^{-1}(A')}(y)$, $\underline{U}_{f^{-1}(A')}(x\text{-}y)$ $=\underline{U}_{A'}(f(x\text{-}y))=\underline{U}_{A'}(f(x)\text{-}f(y))\geq\underline{U}_{A'}(f(x))\wedge\underline{U}_{A'}(f(y))=\overline{U}_{f^{-1}(A')}(x)\wedge\overline{U}_{f^{-1}(A')}(y)$.

Similarly, $\overline{U}_{f^{-1}(A')}(x\cdot y)=\overline{U}_{A'}(f(x\cdot y))=\overline{U}_{A'}(f(x)\cdot f(y))\geq\overline{U}_{A'}(f(x))\wedge\overline{U}_{A'}(f(y))=\overline{U}_{f^{-1}(A')}(x)$ $\wedge\overline{U}_{f^{-1}(A')}(y)$, $\underline{U}_{f^{-1}(A')}(x\cdot y)=\underline{U}_{A'}(f(x\cdot y))=\underline{U}_{A'}(f(x)\cdot f(y))\geq\underline{U}_{A'}(f(x))\wedge\underline{U}_{A'}(f(y))=\overline{U}_{f^{-1}(A')}(x)$ $\wedge\overline{U}_{f^{-1}(A')}(y)$. Thus $f^{-1}(A')$ is a grey subring of R. In addition, for all $x\in$her f, $\overline{U}_{f^{-1}(A')}(x)$

$= \overline{U}_{A'}(0')=1,\ \overline{U}_{f^{-1}(A')}(x)=\underline{U}_{A'}(0')=1.$

(2) Let $x\in f^{-1}(A'_{[0,0]})$, then $f(x)\in A'_{[0,0]}$,therefore $\overline{U}_{A'}(f(x))=1,\ \underline{U}_{A'}(f(x))=1$, that is, $x\in(f^{-1}(A'))_{[0,0]}$. Furthermore, for all $x\in(f^{-1}(A'))_{[0,0]}$, we have $\overline{U}_{f^{-1}(A')}(x)=\underline{U}_{f^{-1}(A')}(x)=1$, consequently $x\in f^{-1}(A'_{[0,0]})$.

(3) If A is constant 1 on her f, then $\overline{U}_{f^{-1}(A)}(f(x))=\overline{U}_{f(A)}(f(x))=\overline{U}_{A}(x)$, $\underline{U}_{f^{-1}(f(A))}(x)=\underline{U}_{f(A)}(f(x))=\overline{U}_{A}(x)$, thus (3) holds.

Lemma 2.3 Let L be a complete lattice, R and R' are two rings , $f: R\to R'$ is an emimorphim, then if A is a grey subring of R, then $f(A)$is a grey subring of R',and furthermore, A is constant 1 on her f ,then $f(A_{[0,0]})=(f(A))_{[0,0]}$.

Proof: Because A is a grey subring, then for all $x', y'\in R'$,

$$\overline{U}_{f(A)}(x'-y')=\vee\{\overline{U}_{A}(z)\mid z\in f^{-1}(x'-y')\}\geq\vee\{\overline{U}_{A}(x-y)\mid x\in f^{-1}(x'), y\in f^{-1}(y')\}$$
$$=(\vee\{\overline{U}_{A}(x)\mid x\in f^{-1}(x')\})\wedge(\vee\{\overline{U}_{A}(y)\mid y\in f^{-1}(y')\})$$
$$=\overline{U}_{f(A)}(x')\wedge\overline{U}_{f(A)}(y'),$$

Similarly,we have

$$\underline{U}_{f(A)}(x'-y')\geq\underline{U}_{f(A)}(x')\wedge\underline{U}_{f(A)}(y'),$$
$$\overline{U}_{f(A)}(x'\cdot y')\geq\overline{U}_{f(A)}(x')\wedge\overline{U}_{f(A)}(y'),$$
$$\underline{U}_{f(A)}(x'\cdot y')\geq\underline{U}_{f(A)}(x')\wedge\underline{U}_{f(A)}(y'),$$

for all $x', y'\in R'$.

By Lemma 2.1, we have $\overline{U}_{f(A)}(0')=\underline{U}_{f(A)}(0')=1$, consequently, $f(A)$ is a grey subring of R'.

If $x'\in f(A))_{[0,0]}$,then $\overline{U}_{f(A)}(x')=\underline{U}_{f(A)}(x')=1$, since f is an emimorphism, so $x\in R$ and $f(x)= x'$. According to Lemma 2.1 we get

$$\overline{U}_{f(A)}(x')=\overline{U}_{f(A)}(f(x))=\overline{U}_{A}(x)=1,$$
$$\underline{U}_{f(A)}(x')=\underline{U}_{f(A)}(f(x))=\underline{U}_{A}(x)=1,$$

thus $x'\in f(A_{[0,0]})$, i. e. , $(f^{-1}(f(A)))_{[0,0]}\subseteq f(A_{[0,0]})$, therefore $f(A_{[0,0]})=(f(A))_{[0,0]}$.

Theorem 2.4 Let R and R' be two rings , $f: R\to R'$ is an emimorphism and L a complete distributive lattice, then there is a one –to-one presperving order correspondence between the grey subring of R and that of R' which are constant 1 on $\ker f$.

Proof: Let φ(R) be the set of all grey subrings of R', φ(R') be the set of all grey subrings of R which is constant 1 on $\ker f$. Let δ: φ(R)→ φ(R') and ε: φ(R')→φ(R) be defined as δ(A)= f (A), and ε(A')= $f^{-1}(A')$.

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