The Partition of All Fuzzy Ideals of Ring R and Their Algebraic Structure

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Abstract The set FI(R) of all fuzzy ideals of a ring R according to the $\mu(0)$ for $\mu \in$ FI(R) is discussed. And then the algebraic structure for the classes and the quotient set from the aspects of lattice,sum $\mu+\eta$, product $\mu \circ \eta$ is studied. The homomorphism and isomorphism relations between this algebraic system are given.

Keywords Fuzzy ideal, Equivalence relation, Class, Quotient set, Semigroup, Lattice

1. The partition of fuzzy ideals

Through this paper, R denotes a ring.

Definition 1.1 Let μ be a fuzzy subset of R. If for any $x, y \in R$,

(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$; (2) $\mu(xy) \geq \mu(x) \wedge \mu(y)$

then μ is called a fuzzy subring of R.

Definition 1.2 Let μ be a fuzzy subring of R. If for any $x, y \in R$,

$$\mu(xy) \geq \mu(x) \vee \mu(y)$$

then μ is called a fuzzy ideal of R.

If μ is a fuzzy subring of R, then $\mu(0) \geq \mu(x), x \in R$. If R has unit element e, then $\mu(e) \geq \mu(x), x \in R, x \neq 0$.

The sign FI(R) denotes the set of all fuzzy ideals of R. In FI(R), the relation $\sim$ is defined as follows: $\mu \sim \eta \Leftrightarrow \mu(0) = \eta(0)$.

It is not difficult to verify that $\sim$ is a equivalence relation of FI(R).

If $FI_t(R) = \{\mu \mid \mu \in FI(R) \quad and \quad \mu(0) = t\}$, then the partition determined by $\sim$ is $\{FI_t(R) \mid t \in [0,1]\}$.

2. Algebraic structure of $FI_t(R)$

In this section, algebraic structure of $FI_t(R)$ is studied from three aspects: addition of fuzzy sets of lattice, ring and multiplication of fuzzy sets of ring.

Theorem 2.1[1] Inclusion relation $\leq$ of $FI(R)$ concerning fuzzy set forms complete lattice. It is denoted by $(FI(R),\vee,\wedge)$.

Definition 2.1[1] Let μ and η be fuzzy sets of R. Addition $\mu+\eta$ is defined as follows: $(\mu+\eta)(x)=\vee\{\mu(y)\wedge\eta(z)\mid y+z=x\},x\in R$.

Theorem 2.2 $FI(R)$ with regard to addition of fuzzy sets form unit semigroup. It is denoted by $(FI_t(R),+)$ and its unit element is $1_{\{0\}}$.

Theorem 2.3[3] $FI_t(R)$ concerning inclusion relation $\leq$ of fuzzy set forms complete modul lattice. It is denoted by $(FI_t(R),\vee,\wedge)$ and $\mu\vee\eta=\mu+\eta$.

Theorem 2.4 $FI_t(R)$ with regard to addition of fuzzy sets form unit semigroup. It is denoted by $(FI_t(R),+)$ and its unit element is $t_{\{0\}}$.

Definition 2.2[2] Let μ and η be fuzzy sets of R. Product $\mu o\eta$ is defined as follows

$$(\mu o\eta)(x)=\vee\{\wedge_{i=1}^{n}(\mu(y_i)\wedge\eta(z_i))\mid n\in N,\sum_{i=1}^{n}y_iz_i=x\}.$$

Theorem 2.5 $FI_t(R)$ with regard to multiplication $\mu o\eta$ of fuzzy sets form semigroup. It is denoted by $(FI_t(R),o)$. If R have unit element ,then $(FI_t(R),o)$ is a unit semigroup and its unit element is $t_{\{e\}}$.

Theorem 2.6 $FI(R)$ with regard to multiplication $\mu o\eta$ of fuzzy sets forms semigroup. It is denoted by $(FI(R),o)$. If R have unit element ,then $(FI(R),o)$ is a unit semigroup and its unit element is $1_{\{e\}}$.

3. Relations of among different equivalent classes

Let μ be a fuzzy subset of R and a a positive real number. If for any $x\in R$, we have $a(\mu(x))\in[0,1]$,then a and η are called multiplied and their product $a\mu$ is defined: $(a\mu)(x)=a(\mu(x)),x\in R$.

Lemma 3.1 Let a be a positive real number and $b_i\in[0,1],i\in I,I$ any subscript set and $ab_i\in[0,1]$. Then $a(\vee_{i\in I}b_i)=\vee_{i\in I}(ab_i)$.

Theorem 3.2 Let $s,t\in[0,1]$. Then

(1) $(FI_s(R),\vee,\wedge)\cong FI_t(R),\vee,\wedge)$, (2) $(FI_s(R),+)\cong(FI_t(R),+)$,

(3) $(FI_s(R),\text{o}) \cong FI_t(R),\text{o})$.

Proof: The map f from $FI_s(R)$ to $FI_t(R)$ is defined as follows:

$$f: FI_s(R) \to FI_t(R),\ \mu \to (t/s)\mu .$$

For $\mu \in FI_s(R)$, it is obvious that t/s and μ are and $(t/s)\mu \in FI_t(R)$. It is not verify that f is a bimorphism from $FI_s(R)$ to $FI_t(R)$.

(1) $\forall\ \mu,\eta \in FI_s(R)$,

$$f(\mu \vee \eta) = (t/s)(\mu \vee \eta) = ((t/s)\mu) \vee ((t/s)\eta) = f(\mu) \vee f(\eta),$$

$$f(\mu \wedge \eta) = (t/s)(\mu \wedge \eta) = ((t/s)\mu) \wedge ((t/s)\eta) = f(\mu) \wedge f(\eta),$$

therefore, $FI_s(R),\vee,\wedge) \cong FI_t(R),\vee,\wedge)$.

(2) $\forall\ \mu,\eta \in FI_s(R),\ x \in R$,

$$\begin{aligned} f(\mu+\eta)(x) &= ((t/s)(\mu+\eta))(x) = (t/s)((\mu+\eta)(x)) \\ &= (t/s)(\vee \{\mu(x_1) \wedge \eta(x_2) \mid x_1 + x_2 = x\}) \\ &= \vee \{((t/s)\mu(x_1)) \wedge ((t/s)\eta(x_2)) \mid x_1 + x_2 = x\}) \\ &= \vee \{((t/s)\mu)(x_1) \wedge ((t/s)\eta)(x_2) \mid x_1 + x_2 = x\}) \\ &= ((t/s)\mu + (t/s)\eta)(x) = (f(\mu) + f(\eta))(x), \end{aligned}$$

therefore, $(FI_s(R),+) \cong (FI_t(R),+)$ and f keeps unit element, that is $f(s_{\{0\}}) = f(t_{\{0\}})$.

(3) $\forall\ \mu,\eta \in FI_s(R),\ x \in R$,

$$\begin{aligned} f(\mu \text{o} \eta)(x) &= ((t/s)(\mu \text{o} \eta))(x) = (t/s)((\mu \text{o} \eta)(x)) \\ &= ((t/s)(\ \vee \{\ \wedge_{i=1}^{n} (\mu(y_i) \wedge \eta(z_i)) \mid n \in N, \sum_{i=1}^{n} y_i z_i = x\}) \\ &= \vee \{(t/s)(\ \wedge_{i=1}^{n} (\mu(y_i) \wedge \eta(z_i))) \mid n \in N, \sum_{i=1}^{n} y_i z_i = x\} \\ &= \vee \{(\ \wedge_{i=1}^{n} ((t/s)\mu(y_i) \wedge ((t/s)\eta(z_i)) \mid n \in N, \sum_{i=1}^{n} y_i z_i = x\} \\ &= ((t/s)\mu) \text{o} ((t/s)\ \eta)(x) = (f(\mu) \text{o} f(\eta))(x), \end{aligned}$$

therefore, $f(\mu \text{o} \eta) = f(\mu) \text{o} f(\eta)$, that is $FI_s(R),\text{o}) \cong FI_t(R),\text{o})$.

If R has unit element, it is not difficult to verify that f keeps one, that is $f(s_{\{e\}}) = f(t_{\{e\}})$.

4 Algebraic structure of quotient set $FI(R)/\sim$

Algebraic operations on $FI(R)/\sim$ are induced by making use of algebraic operations on $FI(R)$ in this section, and the structure of $FI(R)/\sim$ concerning these algebraic operations is discussed.

For $\mu \in FI(R)$, $[\mu]$ denotes equivalent class that μ belongs to with regard of equivalent relation $\sim$.

Theorem 4.1 The following definition are algebraic operations on $FI(R)/\sim$

(1) $[\mu]\vee[\eta]=[\mu\vee\eta]$; (2) $[\mu]\wedge[\eta]=[\mu\wedge\eta]$;

(3) $[\mu]+[\eta]=[\mu+\eta]$; (4) $[\mu]\circ[\eta]=[\mu\circ\eta]$.

Theorem 4.2 $FI(R)/\sim$ forms complete lattice with regard of two operations (1) ,(2) of Theorem 4.1. It is denoted by $(FI(R)/\sim,\vee,\wedge)$, and

$$(FI(R),\vee,\wedge)\sim(FI(R)/\sim,\vee,\wedge).$$

Theorem 4.3 $FI(R)/\sim$ forms unit semigroup with regard of addition of Theorem 4.1 (3). It is denoted by $(FI(R)/\sim,+)$, unit element is and

$$(FI(R),+)\sim(FI(R)/\sim,+).$$

Theorem 4.4 $FI(R)/\sim$ forms semigroup with regard of multiplication of Theorem 4.1(4). It is denoted by $(FI(R)/\sim,\circ)$, and

$$(FI(R),\circ)\sim(FI(R)/\sim,\circ).$$

If R has unit element e, $FI(R)/\sim$ has unit element $[1_{\{e\}}]$ and the homomorphism above keeps unit element.

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