# Multiobjective linear programming with fuzzy cost coefficients

Fang'an Deng, Sanyang Liu
(Department of Applied Mathematics, Xidian University, Xi'an, Shaanxi 710071,China)

**Abstract**

We focus on an MOLP problem with fuzzy-numbered cost coefficients. Base on the membership functions, the problem is transformed into a multiobjective problem with parametrically interval-valued MOLP problem. According to the preference of the decision- maker (DM),a simple method for solving the $\alpha$-efficient extreme solution is presented, and a fuzzy set of the efficient extreme bases is obtained. Finally, the application of this method is illustrated with a numerical example.

**Keywords:** Fuzzy-numbered cost coefficient; parametrically interval values; MOLP

## 1. Introduction

For a multiobjective linear programming (MOLP), how to estimate the exact values of the coefficients is a problematic task. Normally, the coefficients are either given by a decision-maker (DM) subjectively or by statistical inference from historical date. Many authors considered this problem as a fuzzy linear programming (FLP) with fuzzy coefficients of which a membership function was defined for each fuzzy coefficient, and a fuzzy or crisp solution have been obtained.

As regard to a multiple-objective optimization problem, Bitran[1] and Steuer[2] developed different algorithms to solve an MOLP in which the cost coefficients are interval-valued. They applied the vector—maximum theory to find the efficient extreme points. Then, Haiso-Fan Wang and Miao-Liang Wang[3] proposed both theories and solution procedures for an MOLP with interval-valued coefficients, by the proposed tolerance analysis of a nondominated set, a fuzzy set of the efficient extreme bases can be obtained.

In this paper, an approach will be proposed based on Hsiao and Miao's FMOLP methods. This paper is organized in the following way. In section 2,we shall give some definitions, notation and results. Section 3 is the main part of this paper, we expound our approach to solve a FMOLP problem with fuzzy-numbered cost coefficients. In section 4,we give a simple numerical example, which helps to understand our approach.

## 2. Results on an MOLP with fuzzy-numbered cost coefficients

In this study, we shall discuss a FMOLP problem in the following form

$$
\begin{aligned}
Max\overline{Z}(\alpha) &= (\overline{Z}^1(\alpha), \overline{Z}^2(\alpha), \cdots, \overline{Z}^k(\alpha))^T \\
&= (\overline{C}^1 x, \overline{C}^2 x, \cdots, \overline{C}^k x)^T \qquad (1) \\
\text{s.t.} \quad & Ax \le b \\
& x \ge 0
\end{aligned}
$$

where T means "transpose", $A=[a_{ij}]$, for $i=1,2,\cdots,m;\ j=1,2,\cdots,n$ is an $m\times n$ constraint matrix; b=[$b_j$], for I=1,2,⋯, m is the column vector of the right-hand side, $\overline{C^l}=[\overline{C_j^l}]$, j=1,⋯,n; $l=1,\cdots,K$ is a column vector with fuzzy numbers.

Let $B$ be a set of bases. If $\overline{B}$ is the efficient extreme bases of problem (1), then $\overline{B}$ is a fuzzy subset of $B$ defined by

$$\overline{B}=\{(\mu_{\overline{B}}(\beta))\ \ |\ \forall\ \beta\in B\} \qquad (2)$$

where $\mu_{\overline{B}}(\beta)$ is the degree of the membership of $\beta$ in $\overline{B}$.

Let $\mu_j^l$ be the membership function of $c_j^l$ for each j, l, then an $\alpha$-cut of the fuzzy set $\bar{c}_j^l$ is a crisp interval and can be found in the form $[Min_{c\in R}\{c|\mu_j^l(c)\geq\alpha\ \},\ \mathrm{Max}_{c\in R}\{c|\mu_j^l(c)\geq\alpha\ \}]$.

Furthermore, by the convexity of a fuzzy number, the interval can be get as $[\inf\{(\mu_j^l)^{-1}(\alpha)\},$ $\sup\{(\mu_j^l)^{-1}(\alpha)\}]$. Therefore, model(1) can be transformed into a crisp MOLP with the bounds of intervals being functions of $\alpha$ and parametrically interval-valued cost coefficients as defined in the following:

$$\begin{aligned}\text{Max}\quad & z(\alpha)=(z^1(\alpha),z^2(\alpha),\ldots,z^K(\alpha))^T=(c^1(\alpha)x,c^2(\alpha)x\ldots,c^K(\alpha)x)^T\\ \text{s.t.}\quad & Ax\leq b,\\ & x\geq 0\end{aligned} \qquad (3)$$

where $c^l(\alpha)=[c_j^l(\alpha)]$, $c_j^l(\alpha)\in[\inf\{(c_j^l(\alpha))^{-1}\},\sup\{(c_j^l(\alpha))^{-1}\}]$, for l=1,⋯,K, j=1,⋯,n with $\inf\{(c_j^l(\alpha))^{-1}\}$, and $\sup\{(c_j^l(\alpha))^{-1}\}$ being lower and upper bounds of the cost coefficients, respectively.

In fact, once value of $\alpha$ is given, problem (3) becomes an interval-valued problem with constant bounds as denoted by p($\alpha$). In addition, due to the nested cost intervals, for any pair of $\alpha,\beta\in[0,1]$ with $\alpha\leq\beta$, the gradient of p($\beta$) will be an element of p($\alpha$).

Let $B(\alpha_i)$ denote the set of efficient bases of p($\alpha_i$). In particular,an importion result can be restated as

**Theorem1**[3] If $0\leq\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_n\leq 1$, then $B(1)\subseteq B(\alpha_n)\subseteq\cdots\subseteq B(\alpha_1)\subseteq B(0)$.

For problem (3), since there exist infinite gradients,any single objective function will become a multiple or infinite objective functions, and the gradients can be expressed as a convex combination of the $K\times 2^n$ extreme gradients. In order to find out the extreme efficient solution,it is necessary first to find out the criterion cone so as to reduce the number of objectives,then,to find an irreducible generator of the criterion cone by use Telgen's method[3]. However, when K or n becomes larger, it is difficult to obtain the irreducible generator of the criterion cone by using the method. In next section, we will present a simple approach to get the efficient solution of problem (3).

## 3. A simple method with fuzzy –numbered cost coefficients

### 3.1 Notations and definitions

Now, let us restate model (3) as the following forms

$$\text{Max}\quad \begin{pmatrix} z^1(\alpha) \\ z^2(\alpha) \\ \vdots \\ z^K(\alpha) \end{pmatrix} = \begin{pmatrix} [\underline{c_{11}},\overline{c_{11}}] & [\underline{c_{12}},\overline{c_{12}}] & \cdots & [\underline{c_{1n}},\overline{c_{1n}}] \\ [\underline{c_{21}},\overline{c_{21}}] & [\underline{c_{22}},\overline{c_{22}}] & \cdots & [\underline{c_{2n}},\overline{c_{2n}}] \\ \vdots & \vdots & \vdots & \vdots \\ [\underline{c_{K1}},\overline{c_{K1}}] & [\underline{c_{K2}},\overline{c_{K2}}] & \cdots & [\underline{c_{Kn}},\overline{c_{Kn}}] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{4}$$

$$\text{s.t.}\quad Ax \le b, \quad x \ge 0$$

where $[\underline{c_{lj}},\overline{c_{lj}}]$ denotes the j-th cost coefficient of $l$-th objective function, and $\underline{c_{lj}},\overline{c_{lj}}$ are functions of $\alpha$.

We will define an operation of model (4) as follows:

$$Max\, z^l(\alpha) = \sum_{j=1}^{n} [\underline{c_{lj}},\overline{c_{lj}}]x_j = [\sum_{j=1}^{n} \underline{c_{lj}}x_j, \sum_{j=1}^{n} \overline{c_{lj}}x_j] \tag{5}$$

### 3.2 Determination of a $\alpha$-efficient extreme bases by means of a compromise objective function

When $\alpha$ is given, in order to determine a $\alpha$-efficient extreme base of the FMOLP problem (4), we propose a new method as follows.

The simple way of doing this is only to choose the two representatives $\underline{c_{lj}},\overline{c_{lj}}$ of each interval $[\underline{c_{lj}},\overline{c_{lj}}]$,which are lower and upper bounds of cost coefficient, respectively, and instead of solving problem (4),we solve the following LP problem by meansof the usual algorithms:

$$Max\{\underline{z}^l(x) = \sum_{j=1}^{n} \underline{c}_{lj}x_j \,|Ax \le b, x \ge 0\} \qquad (l = 1,2,\cdots,K) \tag{6}$$

$$Max\{\overline{z}^l(x) = \sum_{j=1}^{n} \overline{c}_{lj}x_j |Ax \le b, x \ge 0\} \qquad (l = 1,2\cdots,K) \tag{7}$$

Next, we will consider two extreme cases, which reflect an attitude of the decision maker that is either pessimistic or optimistic, and corresponding models are stated as problems (8) and (9):

$$Max\{z_{Min}(x) = \sum_{j=1}^{n} \min\{\underline{c_{lj}}\}_{l=1}^{K} x_j |Ax \le b, x \ge 0\}, \tag{8}$$

$$Max\{z_{Max}(x) = \sum_{j=1}^{n} \max\{\overline{c}_{lj}\}_{l=1}^{K} x_j |Ax \le b, x \ge 0\}, \tag{9}$$

Based on the method described above, there are 2($K$+$1$) linear optimization problems are included in (6)----(9).

## 4. An example

Consider the following model (10)[3]:

$$
\begin{aligned}
&Max \quad \overline{C}_1^1 x + \overline{C}_2^1 y + \overline{C}_3^1 z \\
&Max \quad \overline{C}_1^2 x + \overline{C}_2^2 y + \overline{C}_3^2 z \\
&Max \quad \overline{C}_1^3 x + \overline{C}_2^3 y + \overline{C}_3^3 z
\end{aligned}
$$

s.t.

$$
\begin{aligned}
&3x - y + 3z \le 6 \\
&-x + 2y - z \le 9 \\
&-x + 5y + 4z \le 8 \\
&5x - z \le 10 \\
&-x - y + 2z \le 8 \\
&x, y, z \ge 0
\end{aligned}
\tag{10}
$$

The functions of these fuzzy coefficients are defined as follows:

$$
\mu_1^1(C_1^1) = \begin{cases} C_1^1 - 5, & C_1^1 \in [5,6]; \\ 1, & C_1^1 \in [6,8]; \\ (10 - C_1^1)/2, & C_1^1 \in [8,10]; \\ 0, & otherwise; \end{cases} \tag{11}
$$

$$
\mu_2^1(C_2^1) = \begin{cases} C_2^1 + 1, & C_2^1 \in [-1,0]; \\ 0, & otherwise; \end{cases} \tag{12}
$$

$$
\mu_3^1(C_3^1) = \begin{cases} (C_3^1 + 2)/0.5, & C_3^1 \in [-2,-1.5] \\ (-1 - C_3^1)/0.5, & C_3^1 \in [-1.5,-1]; \\ 0, & otherwise; \end{cases} \tag{13}
$$

$$
\mu_1^2(C_1^2) = \begin{cases} (C_1^2 - 1)/2, & C_1^2 \in [1,3]; \\ (5 - C_1^2)/2, & C_1^2 \in [3,5]; \\ 0, & otherwise; \end{cases} \tag{14}
$$

$$
\mu_2^2(C_2^2) = \begin{cases} C_2^2 - 3, & C_2^2 \in [3,4]; \\ (6 - C_2^2)/2, & C_2^2 \in [4,6]; \\ 0, & otherwise; \end{cases} \tag{15}
$$

$$
\mu_3^2(C_3^2) = \begin{cases} \sqrt{1 - [(C_3^2 - 1.5)/1.5]^2}, & C_3^2 \in [0,3]; \\ 0, & otherwise; \end{cases} \tag{16}
$$

$$\mu_1^3(C_1^3)=\begin{cases}(C_1^3+1)/2, & C_1^3\in[-1,1];\\(5-C_1^3)/4, & C_1^3\in[1,5];\\0, & otherwise;\end{cases} \tag{17}$$

$$\mu_3^3(C_3^3)=\begin{cases}C_3^3-1, & C_3^3\in[0,1];\\0, & otherwise;\end{cases} \tag{18}$$

4.1 $\alpha$-level transformation

For each $\alpha$-level, we can obtain the corresponding cost intervals. Take $\overline{C}_1^1$ as an example, we have

$$\mu_1^1(C_1^1)\geq\alpha \quad \text{iff} \quad \begin{cases}C_1^1-5\geq\alpha, & C_1^1\in[5,6];\\(10-C_1^1)/2\geq\alpha, & C_1^1\in[8,10];\end{cases} \tag{19}$$

Then, we have $C_1^1\in[5+\alpha,\ 10-2\alpha]$. Similarly, other cost intervals can be obtained. Thus, problem (10) is transformed into the following:

$$\begin{aligned}
&Max \quad [5+\alpha,10-2\alpha]x+[-1+\alpha,\ 0]y+[-2+0.5\alpha,-1-0.5\alpha]z\\
&\text{Max} \quad [1+2\alpha,5-2\alpha]x+[3+\alpha,\ 6-2\alpha]y+[1.5-1.5\sqrt{1-\alpha^2},\ 1.5+1.5\sqrt{1-\alpha^2}]z\\
&Max \quad [-1+2\alpha,\ 5-4\alpha]x+[1+5\alpha,\ 6]z\\
&s.t. \quad 3x-y+3z\leq 6\\
&\qquad -x+2y-z\leq 9\\
&\qquad -x+5y+4z\leq 8\\
&\qquad 5x-z\leq 10\\
&\qquad -x-y+2z\leq 8\\
&\qquad x,y,z\geq 0
\end{aligned} \tag{20}$$

4.2 Finding all possible efficient bases

According to the models (6)------(9), we have p($\alpha$) with respect to $\alpha$ as the following table 1.The results obtained approximately agree with the reference [3]. The largest allowable interval of the efficient extreme bases can be described as follows:

| Basis | largest allowable interval |
|---|---|
| $(0,1.6,0)^T$ | [0,0.5] |
| $(0,0,2)^T$ | [0,1] |
| $(2,0,0)^T$ | [0,1] |
| $(2,2,0)^T$ | [0,1] |
| $(2.09,1.65,0.46)^T$ | [0,1] |

Table 1 all possible efficient for $\alpha = 0,0.5,0.96,1$

| $\alpha$ | (6)efficient extreme solutions | (7) efficient extreme solutions | (8) efficient extreme solutions | (9)efficient extreme solutions | efficient extreme bases |
|---|---|---|---|---|---|
| 0 | $(2,0,0)^T$<br>$(2,2,0)^T$<br>$(0,0,2)^T$ | $(2.09,1.65,0.46)^T$<br>$(2,2,0)^T$ | $(0,0,0)^T$ | $(2.09,1.65,0.46)^T$ | $(0,0,0)^T$<br>$(0,0,2)^T$<br>$(2,0,0)^T$<br>$(2,2,0)^T$<br>$(2.09,1.65,0.46)^T$ |
| 0.5 | $(2,0,0)^T$<br>$(2,2,0)^T$<br>$(2.09,1.65,0.46)^T$ | $(2.09,1.65,0.46)^T$<br>$(2,2,0)^T$<br>$(0,0,2)^T$ | $(0,1.6,0)^T$ | $(2.09,1.65,0.46)^T$ | $(0,1.6,0)^T$<br>$(0,0,2)^T$<br>$(2,0,0)^T$<br>$(2,2,0)^T$<br>$(2.09,1.65,0.46)^T$ |
| 0.96 | $(2,2,0)^T$<br>$(0,0,2)^T$ | $(0,0,2)^T$<br>$(2.09,1.65,0.46)^T$ | $(2,2,0)^T$ | $(2.09,1.65,0.46)^T$ | $(0,0,2)^T$<br>$(2,0,0)^T$<br>$(2,2,0)^T$<br>$(2.09,1.65,0.46)^T$ |
| 1 | $(2,2,0)^T$<br>$(0,0,2)^T$ | $(2.09,1.65,0.46)^T$ | $(2,0,0)^T$ | $(2.09,1.65,0.46)^T$ | $(0,0,2)^T$<br>$(2,0,0)^T$<br>$(2,2,0)^T$<br>$(2.09,1.65,0.46)^T$ |

## 5. Conclusion

In this paper, we focus on an MOLP problem whose cost coefficients can be any types of fuzzy numbers When the membership degree is given as a parameter $\alpha$, the problem can be transformed into an MOLP with parametrically interval-valued cost coefficients and the intervals are functions of $\alpha$ .To our method ,when $\alpha$ is set to zero, all efficient extreme solution can be found .When $\alpha$ is increasing in [0,1],the less is the number of the solutions .Thus , the solution sets are nested with respective to $\alpha$ ,so are the objective values, hence the optimal solutions can be obtained corresponding to different levels .The simple method presented in this paper has the advantage that the solution is more intelligible to the decision maker.


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