The functional representation of α, ε, σ - operators

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In this papers a problem of the functional representation of α, ε, σ-operators is determined as solutions of equations, based on t-norms.

Keyword: t-norm, of α, ε, -operators, minimization operator σ, extended operator.

Introduction. Sanchez E. [1,2] was represented a method for the solution fuzzy relation equations and from which many applications be done. The solutions was based on α, ε, σ - operators.

Recall, for any t-norm T and t-conorm $\perp$, α - operator $x\ \alpha\ y$ and ε operator, $x\ \varepsilon\ y$, defined by: $x\ \alpha\ y = Sup(c \in [0,1] : T(x,c) \le y)$ and $x\ \varepsilon\ y = Inf(d \in [0,1] : \perp(x,d) \ge y)$; $x, y \in [0,1]$.

Example. $x\ \alpha\ y = \begin{cases} 1, x \le y \\ y, x > y \end{cases}$ (1) with $T = T_M = \wedge$,

$x\ \varepsilon\ y = \begin{cases} 0, x \ge y \\ y, x < y \end{cases}$ (2) with $\perp = \perp_M = \vee$

are well known E.Sanchez's α and ε operators.
$\wedge$, $\vee$ stands for min and max.

Note ε is the dual operator in the sense of De Morgan's laws of α, $n(x)\ \alpha\ n(y) = n\ (x \varepsilon\ y)$ (3) with $n(z) = 1 - z$.

In order to study minimal solution related to Sup-min compositions of fuzzy relation equations in [2] was introduced a minimization operator

$x\sigma\ y = \begin{cases} y, x \ge y \\ 0, x < y \end{cases}$ (4)

When f and g are additive generators of an Archimedean t-norm T_A and an Archimedean t-conorm $\perp_A$ respectively, we denoted α, ε as α_A, ε_A and defined [3] by: $x\ \alpha\ y = f^{-1}(f(y) - f(x \vee y))$ (5)

$x\ \varepsilon\ y = g^{-1}(g(y) - g(x \wedge y))$ (6). Obviously, in this case we have also ε_A is the n-dual operators of α_A:

$n(x)\ \alpha_A\ n(y) = n\ (x \varepsilon_A\ y)$ (7)

Example. From 5,6,7 we can get following n-dual operators with $n(x) = 1 - x$

$$x\ \alpha\ y=\begin{cases}1, x\le y\\ y/x, x>y\end{cases} \qquad x\ \varepsilon\ y=\begin{cases}0, x\ge y\\ \dfrac{y-x}{1-x}, x<y\end{cases}$$

$$x\ \alpha\ y=\begin{cases}1, x\le y\\ 1+y-x, x>y\end{cases} \qquad x\ \varepsilon\ y=\begin{cases}0, x\ge y\\ y-x, x<y\end{cases}$$

$$x\ \alpha\ y=\begin{cases}1, x\le y\\ 1-\dfrac{x-y}{x+y-xy}, x>y\end{cases} \qquad x\ \varepsilon\ y=\begin{cases}0, x\ge y\\ \dfrac{y-x}{1-xy}, x<y\end{cases}$$

Results. New we consider the following equations:

$T(a\oplus b,z)=b$ (8) $\qquad \perp(a\otimes b,z)=b$ (9)

in which $\otimes$ $\oplus$ are some operators; here z is solutions which be defined, in particulary, by using α, ε operators.

Let us now point out some useful propositions that can be used to find the solution of the $Sup-(t-norm)$ compositions of fuzzy relation equations.

If we put in the equation (8) $T=T_A$, $\oplus=\vee$, then we have

Proposition 1. $z=a\alpha_A b$ is the solution of the equation $T_A(a\vee b,z)=b$. Conversely, $z=a\vee b$ is the solution of $T_A(a\alpha_A b,z)=b$.

Proof. By the defitions T_A and α_A can be easily proved (5).

If in (9) we put $\perp=\perp_A$, $\otimes=\wedge$ then we have

Proposition 2. $z=a\varepsilon_A b$ is the solution of $\perp_A(a\wedge b,z)=b$. Conversly, $z=a\wedge b$ is the solution of $\perp_A(a\varepsilon_A b,z)=b$.

Proposition 3. If T and $\perp$ are $n-$dual operations then $n(\perp(a\wedge b,z))=T(n(a)\vee n(b),n(z))=n(b)$.

Proof. By (8), (9), according to (3), (7) and $n-$duality of T and $\perp$ we obtain the proof.

Next we considered the exetended α and ε operations which we denoted as α_E,ε_E

Proposition 4. $z=a\alpha_E b=\begin{cases}(b,1], a\le b\\ b, a>b\end{cases}$ is the solution of the equation $T_M(a\vee b,z)=b$.

Proposition 5. $z=a\varepsilon_E b=\begin{cases}[0,b), a\ge b\\ b, a<b\end{cases}$ is the solution of $\perp_M(a\wedge b,z)=b$.

As a corollary of propositions 4.5 we can get (1), (2) replacing respectively z on $z_{\max} = \max_{z \in A} z$; and z on $z_{\min} = \min_{z \in B} z$.

Proposition 6. The solution of the equation $T_M(a \vee b, z_{\max}) = b$ is $z = a\alpha\, b$; the solution of $\perp_M(a \wedge b, z_{\min}) = b$ is $z = a\varepsilon\, b$.

The extended minimization operation σ_E can be represeted as an solution of (9) with $\perp = \perp_M, \otimes = \varepsilon$.

Proposition 7. $z = a\sigma_E b = \begin{cases} b, a \geq b \\ [0,b), a < b \end{cases}$ is the solution of $\perp_M(a\varepsilon\, b, z) = b$.

Proposition 8. $z = a\sigma\, b$ is the solution of $\perp_M(a\varepsilon\, b, z_{\min}) = b$.

Finally we defined γ_A operator which can be used to study minimal solutionrelated to *Sup* − (*t* − *norm*) composition of fuzzy relation equations.

Proposition 9. $z = a\gamma_A\, b$ is the solution of the equation $\perp_M(a\alpha\, b, z) = a\alpha_A\, b$,

where $z = a\gamma_A\, b = \begin{cases} f^{-1}(f(b) - f(a)), a > b \\ 0, a \leq b \end{cases}$

Example. $z = a\gamma_A\, b = \begin{cases} b/a, a > b \\ 0, a \leq b \end{cases}$

References

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