The Intuitionistic Fuzzy Normal Subgroup and its Some Equivalent Propositions

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Abstract In this paper, on the basis of the intuitionistic fuzzy sets introduced by K. Atanassov. We define the intuitionistic fuzzy normal subgroups and the cut - sets of the intuitionistic fuzzy sets, discuss some properties of the intuitionistic fuzzy normal subgroups. Finally, a few equivalent propositions of the fuzzy subgroup constituting an intuitionistic fuzzy normal subgroup are given.

Keywords Intuitionistic fuzzy sets, cut - sets of intuitionistic fuzzy sets, intuitionistic fuzzy normal subgroups.

1 Preliminaries

Definition 1.1[1] Let X be a nonempty classical set. The triad formed as $A = \{\langle x, \mu_A(x), \upsilon_A(x)\rangle \mid x \in X\}$ on is called an intuitionistic fuzzy set on X. Where the functions $\mu_A : X \to [0,1]$ and $\upsilon_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\upsilon_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leqslant \mu_A(x) + \upsilon_A(x) \leqslant 1$ for each $x \in X$.

All of the intuitionistic fuzzy sets on X are written as $IFS[X]$ for short.

Definition 1.2[1] Let X be a nonempty classical set, $A, B \in IFS[X]$, and $A = \{< x, \mu_A(x), \upsilon_A(x) > \mid x \in X\}$, $B = \{< x, \mu_B(x), \upsilon_B(x) > \mid x \in X\}$. Then the orders and their operations are defined as follows.

(1) $A \subseteq B$ iff $\mu_A(x) \leqslant \mu_B(x)$ and $\upsilon_A(x) \geqslant \upsilon_B(x)$ for all $x \in X$;
(2) $A = B$ iff $\mu_A(x) = \mu_B(x)$ and $\upsilon_A(x) = \upsilon_B(x)$ for all $x \in X$;
(3) $A \cap B = \{< x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\upsilon_A(x), \upsilon_B(x)\} > \mid x \in X\}$;
(4) $A \cup B = \{< x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\upsilon_A(x), \upsilon_B(x)\} > \mid x \in X\}$;
(5) $\bar{A} = \{< x, \upsilon_A(x), \mu_A(x) > \mid x \in X\}$;
(6) $\Box A = \{< x, \mu_A(x), 1 - \mu_A(x) > \mid x \in X\}$;
(7) $\Diamond A = \{< x, 1 - \upsilon_A(x), \upsilon_A(x) > \mid x \in X\}$.

Defintion 1.3[1] Let X be an arbitrary nonempty classical set, $\{A_j \mid j \in J\} \subset IFS[X]$

. If $A_j = \{< x, \mu_{A_j}(x), \upsilon_{A_j}(x) > | \ x \in X\}$. We define $\bigcap_{j\in J} A_j = \{< x, \inf_{j\in J} \mu_{A_j}(x), \sup_{j\in J} \upsilon_{A_j}(x) > | \ x \in X\}$.

Definition 1.4[2] Let G be a classical group, the intuitionistic fuzzy subset $A = \{< x, \mu_A(x), \upsilon_A(x) > | \ x \in G\}$ is called an intuitionistic fuzzy group on G , if the following conditions are satisfied.

(1) $\mu_A(xy) \geqslant \min\{\mu_A(x), \mu_A(y)\}$, $\upsilon_A(xy) \leqslant \max\{\upsilon_A(x), \upsilon_A(y)\}$, for all $x, y \in G$;

(2) $\mu_A(x^{-1}) \geqslant \mu_A(x)$, $\upsilon_A(x^{-1}) \leqslant \upsilon_A(x)$, for all $x \in X$.

2 The intuitionistic fuzzy normal subgroup and its properties

Defintion 2.1 Let G be a classical group, $A = \{< x, \mu_A(x), \upsilon_A(x) > | \ x \in G\}$ be an intuitionistic fuzzy set on G , then A is called an intuitionistic fuzzy normal subgroup on G , if

$$\mu_A(xyx^{-1}) \geqslant \mu_A(y),\ \upsilon_A(xyx^{-1}) \leqslant \upsilon_A(y) \text{ , for all } x, y \in G$$

All of the intuitionistic fuzzy normal subgroups on G are denoted as $IFNS[G]$ for short.

Theorem 2.1 Let G be a classical group. If $A, B \in IFNS[G]$, then $A \cap B \in IFNS[G]$.

Proof Let $A = \{< x, \mu_A(x), \upsilon_A(x) > | \ x \in G\}$,

$B = \{< x, \mu_B(x), \upsilon_B(x) > | \ x \in G\}$.

Then $A \cap B = \{< x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\upsilon_A(x), \upsilon_B(x)\} > | \ x \in G\}$

Let
$$\omega_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$
$$\delta_{A\cap B}(x) = \max\{\upsilon_A(x), \upsilon_B(x)\}$$

On the hand, for arbitrary $x, y \in G$, $A, B \in IFNS[G]$, by Definition 2.1, we have

$$\begin{aligned}\omega_{A\cap B}(xyx^{-1}) &= \min\{\mu_A(xyx^{-1}), \mu_B(xyx^{-1})\} \\ &\geqslant \min\{\mu_A(y), \mu_B(y)\} \\ &= \omega_{A\cap B}(y)\end{aligned}$$

On the other hand, we get

$$\delta_{A\cap B}(xyx^{-1}) = \max\{\upsilon_A(xyx^{-1}), \upsilon_B(xyx^{-1})\}$$

$$\leqslant \max\{\upsilon_A(y), \upsilon_B(y)\}$$
$$= \delta_{A\cap B}(y)$$

Therefore, $A \cap B \in IFNS[G]$.

Corollary 2.2 Let $\{A_j \mid j \in J\} \subset IFNS[G]$. Then $\bigcap_{j\in J} A_j \in IFNS[G]$.

Theorem 2.3 Let G be a classical group. If $A \in IFNS[G]$. Then $\Box A \in IFNS[G]$.

Proof Let $A = \{< x, \mu_A(x), \upsilon_A(x) > \mid x \in G\}$. Then

$$\Box A = \{< x, \mu_A(x), 1 - \mu_A(x) > \mid x \in G\}$$

Let $\delta_A(x) = 1 - \mu_A(x), x \in G$

As $A \in IFNS[G]$. From Definition 2.1, for arbitrary $x, y \in G$, we have

$$\mu_A(xyx^{-1}) \geqslant \mu_A(y),\ \upsilon_A(xyx^{-1}) \leqslant \upsilon_A(y)$$

Consequently, $\delta_A(xyx^{-1}) = 1 - \mu_A(xyx^{-1})$

$$\leqslant 1 - \mu_A(y) = \delta_A(y)$$

Thus, $\Box A \in IFNS[G]$.

Theorem 2.4 Let G be a classical group. If $A \in IFNS[G]$, then $\Diamond A \in IFNS[G]$.

Proof Let $A = \{< x, \mu_A(x), \upsilon_A(x) > \mid x \in G\}$. Then

$$\Diamond A = \{< x, 1 - \upsilon_A(x), \upsilon_A(x) > \mid x \in G\}$$

Let $\varphi_A(x) = 1 - \upsilon_A(x),\ x \in G$.

For arbitrary $x, y \in G$, $A \in IFNS[G]$, we have $\upsilon_A(xyx^{-1}) \leqslant \upsilon_A(y)$

Thus $\varphi_A(xyx^{-1}) = 1 - \upsilon_A(xyx^{-1}) \geqslant 1 - \upsilon_A(y) = \varphi_A(y)$.

Hence $\Diamond A \in IFNS[G]$.

Note By $A, B \in IFNS[G]$, we can't derive from $\bar{A} \in IFNS[G], A \cup B \in IFNS[G]$.

3 Main results

Definition 3.1 Let $A \in IFS[X]$, and $A = \{< x, \mu_A(x), \upsilon_A(x) > \mid x \in X\}$ for arbitrary $\lambda \in [0,1]$, we call $A_{\underset{\cdot}{\lambda}} \triangleq \{x \in X \mid \upsilon_A(x) < \lambda < \mu_A(x)\}$ and $A_\lambda \triangleq \{x \in X \mid \upsilon_A(x) \leqslant \lambda \leqslant \mu_A(x)\}$ a strong λ -cut set and a λ -cut set of the intuitionistic fuzzy set A, respectively. Particularly, whenever $\lambda = 0, 1, A_{\underset{\cdot}{\lambda}} = \emptyset$.

Obviously, $A_{\underset{\cdot}{\lambda}}$ and A_λ are all classical sets, and $A_{\underset{\cdot}{\lambda}} \subseteq A_\lambda$.

Theorem 3.1 Let G be a classical group. Then the intuitionistic fuzzy subgroup A is an intuitionistic fuzzy normal subgroup on G iff A_λ is a classical normal subgroup on G for all $\lambda \in [0,1]$.

Proof Necessity. If A is an intuitionistic fuzzy subgroup on G . Then for arbitrary $x, y \in A_\lambda(\neq \emptyset)$, by Definition 3.1, we have $\upsilon_A(x) \leqslant \lambda \leqslant \mu_A(x)$, $\upsilon_A(y) \leqslant \lambda \leqslant \mu_A(y)$.

Thus, combining with Definition 1.4[2], we can obtain that

$$\mu_A(xy) \geqslant \min(\mu_A(x), \mu_A(y)) \geqslant \lambda$$
$$\upsilon_A(xy) \leqslant \max(\upsilon_A(x), \upsilon_A(y)) \leqslant \lambda$$
$$\mu_A(x^{-1}) \geqslant \mu_A(x) \geqslant \lambda, \qquad \upsilon_A(x^{-1}) \leqslant \upsilon_A(x) \leqslant \lambda$$

i.e. $\upsilon_A(xy) \leqslant \lambda \leqslant \mu_A(xy)$, $\quad \upsilon_A(x^{-1}) \leqslant \lambda \leqslant \mu_A(x^{-1})$ for all $x, y \in A_\lambda$.

Consequently, we derive from $xy \in A_\lambda$, $x^{-1} \in A_\lambda$.

Therefore, A_λ is a subgroup on G .

For arbitrary $x \in G$, $a \in A_\lambda \subseteq G$, by $A \in IFNS[G]$, we can get

$\mu_A(xax^{-1}) \geqslant \mu_A(a) \geqslant \lambda$, $\upsilon_A(xax^{-1}) \leqslant \upsilon_A(a) \leqslant \lambda$.

i.e., $\upsilon_A(xax^{-1}) \leqslant \lambda \leqslant \mu_A(xax^{-1})$.

Thus, we have $xax^{-1} \in A_\lambda$.

And so A_λ is a normal subgroup on G .

Sufficiency. If $A_\lambda \neq \emptyset$ for each $\lambda \in [0,1]$, and A_λ is a normal subgroup on G . Then we have

$\mu_A(xyx^{-1}) \geqslant \mu_A(y)$, $\upsilon_A(xyx^{-1}) \leqslant \upsilon_A(y)$, for all $x, y \in G$.

Otherwise, if there exists x_0 *or* $y_0 \in G$ such that

$$\mu_A(x_0y_0x_0^{-1}) < \mu_A(y_0) \text{ or } \upsilon_A(x_0y_0x_0^{-1}) > \upsilon_A(y_0)$$

Take $\lambda_0 = \frac{1}{2}[\mu_A(y_0) + \mu_A(x_0y_0x_0^{-1})]$ or $\lambda_0 = \frac{1}{2}[\upsilon_A(y_0) + \upsilon_A(x_0y_0x_0^{-1})]$

Evidently, $0 \leqslant \lambda_0 \leqslant 1$, and we can infer that

$\upsilon_A(y_0) < \lambda_0 < \mu_A(y_0)$, $\mu_A(x_0y_0x_0^{-1}) < \lambda_0 < \upsilon_A(x_0y_0x_0^{-1})$

Consequently, we have $y_0 \in A_{\lambda_0} \subseteq A_{\lambda_0}$ and $x_0y_0x_0^{-1} \overline{\in} A_{\lambda_0}$.

This contradicts that A_{λ_0} is a normal subgroup on G .

Hence, we get $\mu_A(xyx^{-1}) \geqslant \mu_A(y)$, $\upsilon_A(xyx^{-1}) \leqslant \upsilon_A(y)$ for all $x, y \in G$.

i.e., $A \in IFNS[G]$.

Theorem 3.2 Let A be a fuzzy subgroup on G . Then the following conditions are equivalent:

(1) $A \in IFNS[G]$;

(2) $A(xyx^{-1}) = A(y)$ for all $x, y \in G$;

(3) $A(xy) = A(yx)$ for all $x, y \in G$.

Proof (1)⇒(2). Let $A = \{< x, \mu_A(x), \upsilon_A(x) > \mid x \in G\}$.

As $A \in IFNS[G]$. Then in the light of Definition 2.1, for arbitrary $x, y \in G$, we have

$$\mu_A(xyx^{-1}) \geqslant \mu_A(y),\ \upsilon_A(xyx^{-1}) \leqslant \upsilon_A(y)$$

Thus, taking advantage of the arbitrary property of x , we get

$$\mu_A(x^{-1}yx) = \mu_A(x^{-1}y(x^{-1})^{-1}) \geqslant \mu_A(y)$$

Therefore, $\mu_A(y) = \mu_A(x^{-1}(xy(x^{-1})x) \geqslant \mu_A(xyx^{-1}) \geqslant \mu_A(y)$

i.e. $\mu_A(xyx^{-1}) = \mu_A(y)$

Similarly, we can prove that $\upsilon_A(xyx^{-1}) = \upsilon_A(y)$

Hence, for all $x, y \in G$, $A(xyx^{-1}) = A(y)$ is proved.

(2)⇒(3). Substituting y for yx in (2), we can get (3) easily.

(3)⇒(1). According to $A(yx) = A(xy)$, we obtain

$$A(xyx^{-1}) = A(yxx^{-1}) = A(y) \geqslant A(y)$$

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