FUZZY STOCHASTIC GOAL PROGRAMMING - AN ADDITIVE MODEL

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An additive model to solve Fuzzy Stochastic Goal Programming (FSGP) is formulated. The method uses arithmetic addition to aggregate the fuzzy stochastic goals to construct the relevant decision function. Cardinal and ordinal weights for nonequivalent fuzzy goals are also incorporated in the method. The solution procedure is illustrated with a numerical example.

Keyword: Fuzzy sets, Goal Programming, Fuzzy Stochastic Goal Programming.

1.Introduction

Goal Programming (GP) is a good decision aid in modeling real world decision problems. Goal programming extends linear programming to problems which involve multiple objectives. It is necessary to specify aspiration levels for the objectives and aims to reduce the deviations from aspiration levels. In the case of a problem with nonequivalent goals the weight or priority of the goal is reflected through its deviation variables. Often, in real world problems the aspiration levels and/or priority factors of the DM, and sometimes even the weights to be assigned to the goals, are imprecise in nature. In such situations the use of fuzzy set theory (Zadeh [16]) comes in handy, and M.K. Luhandjula [17][20] used Fuzzy stochastic set theory in linear programming.

In the present work we investigate a particular modeling which is additive (weighted and preemptive) in FSGP by employing the usual addition as an operator to aggregate the fuzzy stochastic goals in conventional GP the simple additive in model for m goals $G_i(x)$ with deviational variables d_i^+, d_i^-, is defined as

Minimize $\Sigma_{i=1}^{m}(d_i^+ + d_i^-)$

Subject to $G_i(x)+d_i^- - d_i^+ = g_i,$

$d_i^+ d_i^- = 0,$

$d_i^+, d_i^-, x \geq 0, \ i=1,2,....m,$

where g_i is the aspiration level of the i-th goal. Here we develop a similar model using fuzzy stochastic expect instead of deviational variables.

We formulate and discuss the simple and weighted additive models in Section 2, and preemptive priority in the additive model in Section 3. The solution procedures are illustrated with numerical examples. Concluding remarks are made in Section 4.

2. The simple and weighed addition mode

2.1.The simple additive model Consider the FSGP problem:

Find x

to satisfy $G_i(X)\theta g_i$, i=1,2,...,m,

subject to $AX\leq b$.

$X\geq 0$,

where X is an n-vector with components x_1, x_2 x_n and $AX\leq b$ are system constraints in vector notation. The symbol 'θ' refers to the fuzzification and stochastic of till aspiration level (i.e., approximately greater than or equal to). The i-th fuzzy stochastic goal $G_i(X)\theta g_i$ in (1) signifies that the DM is satisfied even if less than the g_i, with g_i a random with known distribution, up to certain tolerance limit is attained. A linear membership function μ_i for the i-th fuzzy stochastic goal $G_i(X)\theta g_i$ can be expressed according to Zimmermann [17, 18] as

$$\mu_i(x,g_i)=\begin{cases}0 & if \quad G_i(X)\geq g_i\\ \dfrac{G_i(X)-L_i}{g_i-L_i} & if \quad L_i\leq G_i(X)\leq g_i\\ 0 & if \quad G_i(X)\leq L_i\end{cases} \qquad (2.1)$$

where L_i is the lower tolerance limit for the fuzzy stochastic goal $G_i(X)$. In case of the goal $G_i(X)\theta g_i$ the membership function is defined as

$$\mu_i(x,g_i)=\begin{cases}0 & if \quad G_i(X)\leq g_i\\ \dfrac{U_i-G_i(X)}{U_i-g_i} & if \quad g_i\leq G_i(X)\leq U_i\\ 0 & if \quad G_i(X)\geq U_i\end{cases} \qquad (2.2)$$

where U_i is the upper tolerance limit.

The additive model of the FSGP problem (1) is formulated by adding the membership functions together as

$$\begin{aligned} &Maximize \quad \sum_{i=1}^{m} E[\mu(x,g_i)] \\ &Subject \quad to \quad E[\mu(x,g_i)] \ge \alpha_i \\ &\qquad AX \le b \qquad\qquad (3) \\ &\qquad \mu_i \le 1, \\ &\qquad X,\mu_i \ge 0, \qquad i=1...m \end{aligned}$$

where $E[\mu_i(x,g_i)]$ denotes the expectation of $\mu_i(x,g_i)$,α_i is a satisfactory threshold fixed by the decision maker. This is a single objective optimization problem which can be solved by employing a suitable classical technique. Because the goals are fuzzy stochastic, unlike conventional GP (minimizing the deviations) the fuzzy stochastic decision function consisting of μ_i and α_i's is to be maximized here.

2.2. Numerical example

Let us consider a mathematical model of the problem having 3 fuzzy stochastic goals with 4 variables and 4 system constraints as follow:

Find X to satisfy the following fuzzy stochastic goals:

$$\begin{aligned} &4x_1+x_2+8x_3+x_4\theta g_1 \\ &4x_1+7x_2+6x_3+7x_4\theta g_2 \qquad (4) \\ &x_1-6x_2+5x_3+10x_4\theta g_3 \end{aligned}$$

subject to

$$\begin{aligned} &7x_1+5x_2+3x_3+x_4 \le 98, \\ &7x_1+x_2+x_3+x_4 \le 117, \\ &x_1+x_2+6x_3+6x_4 \le 130, \\ &9x_1+x_2+6x_4 \le 105, \\ &x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

and

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{cases} \begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix} & with \quad probabilities \quad \begin{pmatrix} 0.2 \\ 0.7 \\ 0.3 \end{pmatrix} \\ \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix} & with \quad probabilities \quad \begin{pmatrix} 0.8 \\ 0.3 \\ 0.7 \end{pmatrix} \end{cases}$$

These fuzzy stochastic constraints may be represented by fuzzy sets U_i (i = 1, 2,3) with membership functions

$$\mu(x, g_1^j) = \begin{cases} 0 & if \quad 4x_1 + x_2 + 8x_3 + x_4 \le k_1^j \\ \dfrac{4x_1 + x_2 + 8x_3 + x_4 - k_1^j}{g_1^j - k_1^j} & if \quad k_1^j \le 4x_1 + x_2 + 8x_3 + x_4 \le g_1^j \\ 0 & if \quad 4x_1 + x_2 + 8x_3 + x_4 \ge g_1^j \end{cases}$$

$$\mu(x, g_2^j) = \begin{cases} 0 & if \quad 4x_1 + 7x_2 + 6x_3 + 7x_4 \le k_2^j \\ \dfrac{4x_1 + 7x_2 + 6x_3 + 7x_4 - k_2^j}{g_2^j - k_2^j} & if \quad k_2^j \le 4x_1 + 7x_2 + 6x_3 + 7x_4 \le g_2^j \\ 0 & if \quad 4x_1 + 7x_2 + 6x_3 + 7x_4 \ge g_2^j \end{cases}$$

$$\mu(x, g_3^j) = \begin{cases} 0 & if \quad x_1 - 6x_2 + 5x_3 + 10x_4 \le k_3^j \\ \dfrac{x_1 - 6x_2 + 5x_3 + 10x_4 - k_3^j}{g_3^j - k_3^j} & if \quad k_3^j \le x_1 - 6x_2 + 5x_3 + 10x_4 \le g_3^j \\ 0 & if \quad x_1 - 6x_2 + 5x_3 + 10x_4 \ge g_3^j \end{cases}$$

g_i^j (j=1,2) are values taken by g_i ,k_i^j reasonably fixed by the decider with $k_i^j < g_i^j$ and

$$\max_{j=1,2} k_i^j < \min_{j=1,2} g_i^j$$

Observe that these membership functions express the decider's desire to be as close as possible to g_i^j without exceeding it and an action x such that $\mu_i(x,g_i^j) = 0$ is not attractive because it does not satisfy the fuzzy objective i when g_i takes the value g_i^j

This is mind, one must restrict to

$$x \in \bigcap_{i=1}^{3} suppU_i$$

where supp U_i denotes the support of U_i.

As we have

$$E[\mu_i(x, g_i)] = \sum_{j=1}^{2} p_i^j \mu_i(x, g_i^j)$$

with p_i^j denoting the probability of g_i to take value g_i^j, we have to solve the linear program

$$\begin{cases} \max \quad 0.2\dfrac{4x_1+x_2+8x_3+x_4-k_1^1}{6-k_1^1}+0.8\dfrac{4x_1+x_2+8x_3+x_4-k_1^2}{8-k_1^2} \\ +0.7\dfrac{4x_1+7x_2+6x_3+7x_4-k_2^2}{8-k_2^1}+0.3\dfrac{4x_1+7x_2+6x_3+7x_4-k_2^2}{4-k_2^2} \\ +0.3\dfrac{x_1-6x_2+5x_3+10x_4-k_3^1}{2-k_3^1}+0.7\dfrac{x_1-6x_2+5x_3+10x_4-k_3^2}{1-k_3^2} \\ 0.2\dfrac{4x_1+x_2+8x_3+x_4-k_1^1}{6-k_1^1}+0.8\dfrac{4x_1+x_2+8x_3+x_4-k_1^2}{8-k_1^2} \geq \alpha_1 \\ 0.7\dfrac{4x_1+7x_2+6x_3+7x_4-k_2^2}{8-k_2^1}+0.3\dfrac{4x_1+7x_2+6x_3+7x_4-k_2^2}{4-k_2^2} \geq \alpha_2 \\ 0.3\dfrac{x_1-6x_2+5x_3+10x_4-k_3^1}{2-k_3^1}+0.7\dfrac{x_1-6x_2+5x_3+10x_4-k_3^2}{1-k_3^2} \geq \alpha_3 \\ x_i \geq 0 \qquad i=1..3 \\ x \in \bigcap_{i=1}^{3} suppU_i \end{cases}$$

which can be written

$$\begin{cases} \max \quad 0.2\dfrac{4x_1+x_2+8x_3+x_4-k_1^1}{6-k_1^1}+0.8\dfrac{4x_1+x_2+8x_3+x_4-k_1^2}{8-k_1^2} \\ +0.7\dfrac{4x_1+7x_2+6x_3+7x_4-k_2^2}{8-k_2^1}+0.3\dfrac{4x_1+7x_2+6x_3+7x_4-k_2^2}{4-k_2^2} \\ +0.3\dfrac{x_1-6x_2+5x_3+10x_4-k_3^1}{2-k_3^1}+0.7\dfrac{x_1-6x_2+5x_3+10x_4-k_3^2}{1-k_3^2} \\ 0.2\dfrac{4x_1+x_2+8x_3+x_4-k_1^1}{6-k_1^1}+0.8\dfrac{4x_1+x_2+8x_3+x_4-k_1^2}{8-k_1^2} \geq \alpha_1 \\ 0.7\dfrac{4x_1+7x_2+6x_3+7x_4-k_2^2}{8-k_2^1}+0.3\dfrac{4x_1+7x_2+6x_3+7x_4-k_2^2}{4-k_2^2} \geq \alpha_2 \\ 0.3\dfrac{x_1-6x_2+5x_3+10x_4-k_3^1}{2-k_3^1}+0.7\dfrac{x_1-6x_2+5x_3+10x_4-k_3^2}{1-k_3^2} \geq \alpha_3 \\ \max_{j=1..2} k_1^j \leq 4x_1+x_2+8x_3+x_4 \leq \min_{j=1,2} g_1^j \\ \max_{j=1..2} k_2^j \leq 4x_1+7x_2+6x_3+7x_4 \leq \min_{j=1,2} g_2^j \\ \max_{j=1..2} k_3^j \leq x_1-6x_2+5x_3+10x_4 \leq \min_{j=1,2} g_3^j \\ x_i \geq 0 \qquad i=1..3 \end{cases}$$

$$if \quad k_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad k_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad k_3 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \quad \alpha = \begin{pmatrix} 0.25 \\ 0.1 \\ 0.27 \end{pmatrix}$$

This problem is solved by using the simplex method and the results obtained are

$x_1=1.0004$, $x_2=0$, $x_3=0.0007$, $x_4=0$,

with achieved goal values and desider's α

$G_1=4.0060$, $G_2=4.0052$, $G_3=1.0067$

2.3. *The weighted additive model*

The weighted additive model is widely used in *GP* and multi-objective optimization techniques to reflect the relative importance of the goals/objectives. In this approach the DM assigns differential weights as coefficients of the individual terms in the simple additive fuzzy stochastic achievement function to reflect their relative importance, i.e., the objective function is formulated by multiplying each membership of the fuzzy stochastic goal with a suitable weight and then adding them together. This leads to the following formulation corresponding to (3).

$$\begin{aligned} &\text{Maximize} \quad \sum_{i=1}^{m} \omega_i E[\mu(x, g_i)] \\ &\text{Subject to} \quad E[\mu(x, g_i)] \geq \alpha_i \\ &\qquad AX \leq b \qquad\qquad (6) \\ &\qquad \mu_i \leq 1, \\ &\qquad X, \mu_i \geq 0, \qquad i = 1..m \\ &\qquad \mu_i = \frac{G_i(x) - L_i}{g_i - L_i} \end{aligned}$$

where ω_i is the relative weight of the i-th fuzzy stochastic goal expectation.

The major difficulty of this method is the DM's task to assess the relative importance of the goals correctly. The phrase 'relative importance' is a fuzzy stochastic concept whose various levels can be stated only imprecisely. However, there are some good approaches in the literature to assess these weights. We may mention in this regard the eigenvector method of Saaty [12], a geometric averaging procedure for constructing super-transitive approximations to binary comparison matrices by Narasimhan [10] the entropy method of Jaynes [7] and the weighted least squares method of Chu et al. [7]. These can be used to suitably specify the weight.

2.4. *Numerical example*

The example which was considered in the previous section is reformulated using relative weights ω=(0.49, 0.31, 0.20). Therefore the objective function (5a) of problem (5) becomes

$$\begin{aligned} &Maximize \quad 0.49E[\mu(x,g_1)]+0.31E[\mu(x,g_2)]+0.20E[\mu(x,g_3)] \\ &Subject \quad to \quad E[\mu(x,g_i)]\geq\alpha_i \\ &\qquad AX\leq b \qquad\qquad (6) \\ &\qquad \mu_i\leq 1, \\ &\qquad X,\mu_i\geq 0, \quad i=1..m \end{aligned}$$

The results obtained with the same constraints (5b) as in (5) are

x_1=1.0043,x_2=0.0051,x_3=1.0479,x_4=0

with achieved goal values

G_1 = 4.0073. G_2 = 4.0059. G_3=4.0493,

It may be noted that, as compared to the previous solution of the simple additive model (which correspond to equal importance of goals), in the present formulation the achievements of G_1 and G_3 (μ_1 and μ_3) have increased, and those of the remaining goals have decreased according to the weighting structure.

3.Preemptive priority in the additive model

In many decision problem, the goals are not commensurable (not in the same measurable unit). Further, sometimes the goals are such that unless a particular goal or a subset of goals is achieved, the other goals should not be considered. In such situations the weighting scheme of the previous section is not an appropriate method, The preemptive priority structure may be stated as $p_i>>p_{i+1}$ which means that the goals in the i-th priority level have higher priority than the goals in the (i + 1)-th priority level.

For the present investigation the problem is subdivided into *k* subproblems, where *k is* the number of priority levels. In the first subproblem the fuzzy stochastic goals belonging to the first priority level have only been considered and solved using the simple additive model as described in Section 1. But at other priority levels the membership values achieved earlier for higher priority levels are imposed as additional constraints. In general the i-th subproblem becomes

$$
\begin{aligned}
&\textit{Maximize} \quad \sum_i^n E[\mu(x,g_i)_{p_i}] \\
&\textit{Subject} \quad \textit{to} \quad E[\mu(x,g_i)_{p_i}] \ge \alpha_i \\
&AX \le b \\
&(\mu)_{p_r} = (\mu^*)_{p_r}, r = 1,2,...j-1 \\
&\mu_s = \frac{G_s - L_s}{g_s - L_s} \\
&\mu_i \le 1, \\
&X, \mu_i \ge 0, \qquad i = 1...m
\end{aligned}
$$

where $(\mu_s)_p$ refers to the membership functions of the goals in the i-th priority level and $(\mu^*)_p$ is the achieved membership value in the r-th ($r \le j - 1$) priority level.

3.1. *Numerical example*

Let the example discussed in Section 1 with three fuzzy stochastic goals have the following two priority levels:

priority level 1: G_1, priority level 2: G_3. priority level 3: G_2

The subproblems are formulated accordingly as defined above. The solution found for the first two subproblems are $\mu_1 = 1$ and $\mu_2 = 0.795$. Now the 3rd and last subproblem to solve is

maximize $E(\mu) = E(\mu_2)$

subject to $\mu_1 = 1$,

$\mu_3 = 0.795$,

$\mu_2 \le 1$

$E(\mu_2) \ge 0.1$

and (5b) excluding $\mu_i \le 1$.

The results are

$x_1=1.0017$, $x_2=0.0009$, $x_3=0.0003$, $x_4=0.0071$, with achieved goal value

$G_1=1.16$, $G_2=-2.03$, $G_3= 1.74$,

Hence, the first priority goals G_1 and G_3 are achieved fully whereas the 2nd priority goal G_2 are achieved partially.

4. Concluding remarks

In this paper, we have illustrated the additive model for solving FSGP problems. The linearity of the model was ensured by its additive property. It is difficult to assign the aspiration values for the membership functions when the corresponding goals themselves are fuzzy stochastic. This difficulty is not present in our model.

The differential weights (cardinal weights) for differing fuzzy stochastic goals within the same priority level can also be used as in conventional GP. In Section 3, while employing preemptive priority in FSGP the achievement values (membership values) of the higher priority goals are preserved when considering the lower priority levels. This is contrary to preserving the achieved goal values (aspiration values) as in conventional GP. The numerical example considered in this paper is only for illustrating the procedure. Case studies and nonlinearity in FSGP are still open for investigation.

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