On Fuzzy Probability

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Abstract: The probability of fuzzy events is redefined. Basic Properties for the probability are discussed. It is given that relation between new probability and old one defined by expectation method.

Key words: Fuzzy event, Probability of fuzzy event, expansion method, expectation method.

1 Introduction

For the probability of fuzzy events[1] , two different definitions have been presented. One is so-called expectation method, and other is so-called expansion method[2]. The latter is narrated to be for fuzzy event A, its fuzzy probability P(A) is defined to be

$$P(A)=\bigvee_{\lambda\in[0,1]}(\lambda\wedge P(A_\lambda)) \qquad (1)$$

The definition has two characteristics, one is considering the stratified structure of fuzzy event, using probability of crisp event A_λ to approximate that of fuzzy event A and, other is comprehensively considering with a conservative point of view that membership level λ of fuzzy event A and corresponding probability value P(A_λ). However, the definition has a clear shortcoming, shown, as example 1.1, that is, the measure for possibility which fuzzy event will happen is too rough.

Example 1.1 Let ζ be a random variable ,and it takes values 1,2,3,4,5, and its distribution series is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0.2 & 0.1 & 0.3 & 0.3 & 0.1 \end{bmatrix}$$

Put Ω={1,2,3,4,5}. Two fuzzy events A and B on Ω are definition as follows:

A=1/1+0.3/2+0.7/3+0.8/4+0.4/5,

B=1/1+0.7/2+0.7/3+0.7/4+0.7/5,

We get, by formula (1), the probabilities of A and B: P(A)=P(B)=0.7.

Clearly, A and B are different very much, and yet their probabilities are same. It is irrational. We think that is caused by $\lambda \wedge P(A_\lambda)$ in formula (1). The operator "$\wedge$" Has imperceptibly lost some information as only one of both λ and $P(A_\lambda)$ is taken. In addition, $\lambda \wedge P(A_\lambda)$ is too slow in reacting the probability of crisp event A_λ, for example, in Example 1.1, take λ=0.7, then A_λ={1,3,4} and $B_\lambda=\Omega$, and so $P(A_\lambda)$=0.8 and $P(B_\lambda)$=1. But $\lambda \wedge P(A_\lambda)=0.7=\lambda \wedge P(B_\lambda)$.

The purpose of this paper is to correct formula (1) and, put forward the new formula which transform $\lambda \wedge P(A_\lambda)$ into $\lambda P(A_\lambda)$. Basic properties for this new fuzzy probability are discussed, and the relation is exposed between new fuzzy probability and old that defined by expectation method.

2 Probability for fuzzy event

Definition 2.1 Let Ω be a sample space, and F(Ω) a set consisting of all fuzzy sets on Ω. $\Psi \subset F(\Omega)$ is called a fuzzy event fidled on Ω ,if

(1) $1_\Omega \in \Psi$,where $1_\Omega(x) \equiv 1, x \in \Omega$;

(2) If $A \in \Psi$, then $A' \in \Psi$, there $A'(x)=1-A(x), x \in \Omega$.

(3) If $A_n \in \Psi$,i=1,2,…,n, $\bigvee_{n=1}^{\infty} A_n \in \Psi$.

Example 2.2 Let $\tau \subset \Omega$ be a crisp event field . Put $\chi_\tau=\{\chi_B : B \in \tau\}$, where, χ_B is The characteristic function of B. Then χ_τ is a fuzzy event field on Ω.

Example 2.3 All constant value fuzzy sets on Ω forms a fuzzy event field on

Ω.

Proposition 2.4 Let τ be a crisp event field on Ω, then there exists a fuzzy event field F(τ) on Ω such that for each $\lambda \in [0,1]$ and every A$\in$ F(τ),we have $A_\lambda \in \tau$.

Proof From example 2.2 we see that χ_τ = F(τ) is the fuzzy event field on Ω. For each $\lambda \in [0,1]$ and every A$\in$ F(τ),there is B$\in \tau$ such that A $=\chi_B$. Clearly, $A_\lambda = \Omega$ for λ=0, and A_λ=B$\in \tau$ for λ>0. □

Remark 2.5 In the sequel, F(τ) is the fuzzy event field having the meaning of proposition 2.4. In addition, for A,B$\in$ F(τ), A$\vee$B is written as A+B if A$\wedge$B=0_Ω; Sometimes A$\wedge$B is written as AB.

Now we define probability of fuzzy event.

Definition 2.6 Let Ω be a sample space, and τ a crisp event field, and P Probability on τ. For each A$\in$F(τ),its probability is defined to be

$$P^*(A) = \bigvee_{\lambda \in I} \lambda P(A_\lambda)$$

Where, I is real unit interval, and the same below.

Example 2.7 For fuzzy events A, B shown as Example 1.1 we have P*(A)=0.56 and P^*(B)=0.7.

Now we discuss basic properties for P^*

Clearly we have

Theorem 2.8 For F(τ) we have

(1) For each A$\in$F(τ),0$\le$ P(A)$\le$1;

(2) $P^*(1_\Omega)$=1, $P^*(0_\Omega)$=0.

Theorem 2.9 Let $A_1, A_2, \cdots, A_n \in$F($\tau$),and $A_i \wedge A_j = 0_\Omega$ (i$\neq$j). Then

$$P^*(A_1 + \cdots + A_n) \le P^*(A_1) + \cdots + P^*(A_n)$$

Proof For each $\lambda \in I$,from the properties of λ-cuts for fuzzy set we have $(A_1 + \cdots + A_n)_\lambda = (A_1 \vee \cdots \vee A_n)_\lambda = (A_1)_\lambda \cup \cdots \cup (A_n)_\lambda$ Since $A_i \wedge A_j = 0_\Omega$ (i$\neq$j),

.$(A_i \wedge A_j)_\lambda=(A_i)_\lambda \cap (A_j)_\lambda=\varnothing$. Thus, it follows from the properties for crisp probability that

$$P^*(A_1+\cdots+A_n)=\vee_{\lambda\in I} \lambda P((A_1+\cdots+A_n)_\lambda)$$

$$=\vee_{\lambda\in I} \lambda P((A_1)_\lambda \cup\cdots\cup (A_n)_\lambda)$$

$$=\vee_{\lambda\in I} \lambda P((A_1)_\lambda+\cdots+ P(A_n)_\lambda)$$

$$=\vee_{\lambda\in I} \lambda P((A_1)_\lambda)+\cdots+\vee_{\lambda\in I} \lambda P((A_n)_\lambda)$$

$$=P^*(A_1)+\cdots+P^*(A_n). \quad \square$$

Example 2.10 Let $\Omega=\{x, y\}$,$P(x)=P(y)=\frac{1}{2}$.

Fuzzy events A,B on Ω are defined as follows:

$$A=1/x+0/y, B=0/x+0.1/y.$$

The $A\vee B=1/x+0.1/y$, $A\wedge B=0_\Omega$, and thus $P^*(A\vee B)=P^*(A+B)=\frac{1}{2}$,

$P^*(A)=\frac{1}{2}$,$P^*(B)=1/20$. Hence $P^*(A+B)\neq P^*(A)+P^*(B)$.

Theorem 2.11 Let A, $B\in F(\tau)$,and AB ,$P^*(A) \geq P^*(B)$. Then

$$P^*(A_1\vee\cdots\vee A_n)\leq P^*(A_1)+\cdots+P^*(A_n)$$

Proof It is easy.

Theorem 2.12 Let $A_1, A_2,\ldots, A_n \in F(\tau)$,Then

$$P^*(A_1\vee\cdots\vee A_n)\leq P^*(A_1)+\cdots+P^*(A_n)$$

Proof $P^*(A_1\vee\cdots\vee A_n)=\vee_{\lambda\in I} \lambda P((A_1)_\lambda)\cup\cdots\cup (A_n)_\lambda)$

$$\leq \vee_{\lambda\in I} \lambda (P((A_1)_\lambda)+\cdots+P((A_n)_\lambda)$$

$$\leq \vee_{\lambda\in I} \lambda (P((A_1)_\lambda)+\cdots+\vee_{\lambda\in I} \lambda P((A_n)_\lambda)$$

$$=P^*(A_1)+\cdots+P^*(A_n). \quad \square$$

3 The relation between new and old fuzzy probability

For fuzzy event A on $\Omega=\{x_1,\cdots,x_n\}$, its probability defined by expectation method is

$$P_*(\mathrm{A})=\sum_{i=1}^{n} A(x_i)P(x_i) \qquad (3)$$

Where, P(x_i) and A(x_i) are , respectively ,the probability and membership grade of x_i.

Theorem 3.1 Let $\Omega=\{x_1,\cdots,x_n\}$ and $\mathrm{A}\in \mathrm{F}(\tau)$. Then,

$\mathrm{P}^*(\mathrm{A})\le P_*(\mathrm{A})$;

(1) $\mathrm{P}^*(\mathrm{A})=P_*(\mathrm{A})$ iff A is constant value fuzzy set .

Proof (1) There is no harm in assuming $\mathrm{A}(x_i)=\lambda_i$,i=1,2,…,n, and $\lambda_1<\lambda_2<\cdots<\lambda_n$.Then

$$A_{\lambda_1}=\Omega, A_{\lambda_2}=\{x_2\cdots,x_n\},\ldots, A_{\lambda_n}=\{\ x_n\}.$$

From $\lambda_1<\lambda_2<\cdots<\lambda_n$ we get

$$\lambda_1 P(A_{\lambda_1})=\lambda_1(P(x_1)+\cdots+P(x_n))\le\lambda_1 P(x_1)+\cdots+\lambda_n P(x_n),$$

$$\lambda_2 P(A_{\lambda_2})=\lambda_2(P(x_2)+\cdots+P(x_n))\le\lambda_2 P(x_2)+\cdots+\lambda_n P(x_n),$$

$$\lambda_3 P(A_{\lambda_3})=\lambda_3(P(x_3)+\cdots+P(x_n))\le\lambda_3(P(x_3)+\cdots+\lambda_n P(x_n).$$

……

$$\lambda_n P(A_{\lambda n})=\lambda_n P(x_n).$$

Thus $\lambda_n P(A_{\lambda n})\le\lambda_{n-1}P(A_{\lambda_{n-1}})\le\cdots\le\lambda_1 P(A_{\lambda_1})$

Hence $\mathrm{P}^*(\mathrm{A})=\underset{\lambda\in I}{\vee}\lambda\,\mathrm{P}(A_\lambda)=\lambda_1 P(A_{\lambda_1})\vee\lambda_2 P(A_{\lambda_2})\vee\cdots\vee\lambda_n P(A_{\lambda n})$

$$=\lambda_1 P(A_{\lambda_1})\le\lambda_1(P(x_1)+\lambda_2(P(x_2)+\cdots+\lambda_n P(x_n)=P_*(A)$$

(2) If A is constant value fuzzy set , then there is $r\in(0,1]$ such that

$\mathrm{A}(x_i)\equiv r, x_i\in\Omega$,i=1,2,…,n. Thus

$\mathrm{P}\,(\mathrm{A})=\underset{\lambda\in I}{\vee}\lambda\,\mathrm{P}(A_\lambda)=r\ (\mathrm{P}(x_1)+P(x_2)+P(x_n)=r$

$$P^*(A)=\sum_{i=1}^{n} A(x_i)P(x_i) = \sum_{i=1}^{n} rP(x_i) = r$$

Hence $P^*(A)=P_*(A)$.

Conversely, if $P^*(A)=P_*(A)$, then A must be constant value fuzzy event .In fact, if A is not constant, then there is no harm in assuming $A(x_i)=\lambda_i$ and $\lambda_1<\lambda_2<\cdots<\lambda_n$. Thus

$$P^*(A)= \lambda_1 P(A_{\lambda_1}) \vee \lambda_2 P(A_{\lambda_2}) \vee \cdots \vee \lambda_n P(A_{\lambda n}).$$

Let $\lambda_i P(A_{\lambda_i}) = \max\{\lambda_1 P(A_{\lambda_1}),\ldots, \lambda_n P(A_{\lambda n})\}$, then

$$P^*(A)=\lambda_i P(A_{\lambda_i}) = \lambda_i (P(x_i)+P(x_{i+1})+ \cdots + P(x_n))$$

$$=\lambda_i (P(x_i)+\lambda_i P(x_{i+1})+\cdots+\lambda_i P(x_n),$$

$$P^*(A)= A(x_1)P(x_1)+A(x_2)P(x_2)+ \cdots + A(x_n)P(x_n)$$

$$=\lambda_1 (P(x_1) +\lambda_2 (P(x_2)+\cdots+\lambda_n P(x_n))$$

It follows from $P^*(A)=P_*(A)$ that

$$\lambda_1 P(x_1)+\lambda_2 P(x_2)+\cdots+\lambda_{i-1} P(x_{i-1})$$

$$=(\lambda_i - \lambda_{i+1})P(x_{i+1})+(\lambda_i - \lambda_{i+2})P(x_{i+2})+\cdots(\lambda_i - \lambda_n)P(x_n)$$

From $\lambda_i<\lambda_{i+1}<\cdots<\lambda_n$ we have

$$\lambda_1 (P(x_1) +\lambda_2 (P(x_2)+\cdots+\lambda_{i-1} P(x_{i-1})<0.$$

This is impossible A is constant value fuzzy event. □

References

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