Fuzzifying Rings Based on Complete Residuated Lattice-Valued Logic

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Abstract: In this paper, we introduce the concept of fuzzifying ring based on complete residuated lattice-valued logic and investigate some of the their algebraic properties.

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Ying [2] used a semantic method of complete residuated lattice-valued logic to propose the concept of fuzzifying topology . Shen [3] further established the theory of fuzzifying groups based on complete residuated lattice-valued logic . In this paper , we are going to discussed fuzzifying ring based on complete residuated lattice-valued logic . For a residuated lattice , we refer to [5] , let £ =<L,+,·,⊗,∝> be a complete residuated lattice, whose least and greatest elements are 0 and 1, respectively , and ⊗ and ∝ are the two binary operations on L such that <L,⊗,∝> is a residuated lattice . For any a , b∈L, we write $a\ \beta\ b=(a\propto b)\cdot(b\propto a)$. In L-valued logic , the set of truth values is L, the only designated truth value being 1 . In other words , a formula φ is valid , we write ⊨φ,if and only if [φ]=1 for every interpretation. The truth valuation rules of predicate logical and set theoretical formulae are displayed as follows .

1 . $[a]=a(a\in L)$, [φ ∨ ψ]=[φ]+[ψ] , [φ ∧ ψ]=[φ]·[ψ] ,[φ $\dot\wedge$ ψ]=[φ]⊗[ψ] , and [φ → ψ]=[φ]∝[ψ] .

2. If X is the universe , then

$$[(\exists x)\varphi(x)]=\sum_{x\in X}[\varphi(x)],\quad [(\forall x)\varphi(x)]=\prod_{x\in X}[\varphi(x)]$$

3. $[x\in A]=A(x)$

In addition, the following derived formulae are necessary,

a. ¬ φ:=φ→0,φ↔ψ:=(φ→ψ)∧(ψ→φ).

b. $A\subseteq B:=(\forall x)((x\in A)\to(x\in B)), A\equiv B:=(A\subseteq B)\wedge(B\subseteq A)$.

In this paper, $\mathcal{F}_L(R)$ denote the L − fuzzy power of R, where R is an arbitrary ring.

Definition 1 . Let $A\in\mathcal{F}_L(R)$, we set

$$r_1(A):=(\forall x)(\forall y)((x\in A)\wedge(y\in A)\to(x+y\in A)$$
$$r_2(A):=(\forall x)((x\in A)\to(-x\in A))$$
$$r_3(A):=(\forall x)(\forall y)((x\in A)\wedge(y\in A)\to(xy\in A))$$

The unary fuzzy predicate $r \in \mathcal{F}_L(\mathcal{F}_L(R))$, is interpreted as the fuzzy property to an L- fuzzifying subring in R, is given as follows:

$$r(A) := r_1(A) \wedge r_2(A) \wedge r_3(A).$$

Theorem 1 . For any $A \in \mathcal{F}_L(R)$,

(1). $\models r_2(A) \to (r_1(A) \leftrightarrow r_4(A))$. (2). $\models r_4(A) \to (\forall x)(x \in A \to o \in A)$.

Where $r_4(A) := (\forall x)(\forall y)((x \in A) \wedge (y \in A) \to (z - y \in A))$

Proof:

(1)$[r_1(A) \leftrightarrow r_4(A)]$

$= \prod_{x,y \in R} (A(x) \cdot A(y) \propto A(x+y)) \ \beta \ \prod_{x,y \in R} (A(x) \cdot A(y) \propto A(x-y))$

$= \prod_{x,y \in R} (A(x) \cdot A(y) \propto A(x+y)) \ \beta \ \prod_{x,y \in R} (A(x) \cdot A(-y) \propto A(x+y))$

$\geqslant \prod_{x,y \in R} (A(y) \ \beta \ A(-y))$

$= \prod_{y \in R} (A(y) \ \beta \ A(-y))$

$= [r_2(A)]$

(2)$[r_4(A)] = \prod_{x,y \in R} (A(x) \cdot A(y) \propto A(x-y))$

$\leqslant \prod_{x \in R} (A(x) \propto A(o)) = [(\forall x)(x \in A) \to (o \in A))]$

Definition 2. Let $A \in \mathcal{F}_L(R)$, we set

$$I_L(A) := (\forall x)(\forall y)((y \in A) \to (xy \in A)),$$

$$I_R(A) := (\forall x)(\forall y)((x \in A) \to (xy \in A)).$$

The unary fuzzy predicate $I \in \mathcal{F}_L(\mathcal{F}_L(R))$, called fuzzy ideal , is given as follows :

$$I(A) := r(A) \wedge I_L(A) \wedge I_R(A)$$

Theorem 2 . Let $r_5(A) := (\forall x)(\forall y)((x \in A) \vee (y \in A) \to (xy \in A))$ then

$$\models r_5(A) :\to I_L(A) \wedge I_R(A)$$

Proof : $[r_5(A)] = \prod_{x,y \in R} ((A(x) + A(y)) \propto A(xy)) \leqslant \prod_{x,y \in R} (A(y) \propto A(xy)) = [I_L(A)]$, and

$[r_5(A)] = \prod_{x,y \in R} ((A(x) + A(y)) \propto A(xy)) \leqslant \prod_{x,y \in R} (A(x) \propto A(xy)) = [I_R(A)]$

Corollary 1 $\models r_5(A) \wedge r(A) \to I(A)$

Theorem 3 Let $A \in \mathcal{F}_L(R)$, $B \in \mathcal{F}_L(H)$, and let $f : R \to H$ be a surjective ring homomorphic . Then

1 . $\models r(A) \to r(f(A))$; $\models I(A) \to I(f(A))$

2 . $\models r(B) \leftrightarrow r(f^{-1}(B))$; $\models I(B) \leftrightarrow I(f^{-1}(B))$

Proof : 1 . $[r_1(f(A))] = \prod_{z,w\in H}(f(A)(z)\cdot f(A)(w) \propto f(A)(z+w))$

$= \prod_{z,w\in H}((\sum_{x\in f^{-1}(z)} A(x)\cdot \sum_{y\in f^{-1}(w)} A(y)) \propto \sum_{\substack{x\in f^{-1}(z),\\ y\in f^{-1}(w)}} A(x+y))$

$\geqslant \prod_{z,w\in H} \prod_{\substack{x\in f^{-1}(z),\\ y\in f^{-1}(w)}} (A(x)\cdot A(y) \propto A(x+y))$

$= \prod_{x,y\in G}(A(x)\cdot A(y) \propto A(x+y))$

$=[r_1(A)]$

Similarly , $[r_3(f(A))] \geqslant [r_3(A)]$

$[r_2(f(A))] = \prod_{z\in H}(f(A)(z) \propto f(A)(-z))$

$= \prod_{z\in H}((\sum_{x\in f^{-1}(z)} A(x) \propto \sum_{(-x)\in f^{-1}(-z)=(-f^{-1}(z))} A(-x))$

$\geqslant \prod_{z\in H} \prod_{x\in f^{-1}(z)} (A(x) \propto A(-x))$

$= \prod_{x\in R}(A(x) \propto A(-x))$

$=[r_2(A)]$

$[I_L(f(A))] = \prod_{z,w\in H}(f(A)(w) \propto f(A)(zw))$

$= \prod_{z,w\in H}(\sum_{y\in f^{-1}(w)} A(y) \propto \sum_{\substack{x\in f^{-1}(z)\\ y\in f^{-1}(w)}} A(xy))$

$\geqslant \prod_{z,w\in H} \prod_{\substack{x\in f^{-1}(z)\\ y\in f^{-1}(w)}} (A(y) \propto A(xy))$

$= \prod_{x,y\in R}(A(y) \propto A(xy))$

$=[I_L(A)]$

Similarly, $[I_R(f(A))] \geqslant [I_R(A)]$.

Therefore, we have $[r(f(A))] \geqslant [r(A)]$, $[I(f(A))] \geqslant [I(A)]$.

2. $[r_1(f^{-1}(B))] = \prod_{x,y\in R}(f^{-1}(B)(x)\cdot f^{-1}(B)(y) \propto f^{-1}(B)(x+y))$

$= \prod_{x,y\in R}(B(f(x))\cdot B(f(y)) \propto B(f(x+y)))$

$= \prod_{x,y\in R}(B(f(x))\cdot B(f(y)) \propto B(f(x)+f(y)))$

$= \prod_{z,w\in H}(B(z)\cdot B(w) \propto B(z+w))$

$=[r_1(B)]$.

where $f(x)=z, f(y)=w$.

Similarly, $[r_2(f^{-1}(B))]=[r_2(B)]$, $[r_3(f^{-1}(B))]=[r_3(B)]$.

$$[I_L(f^{-1}(B))]=\prod_{x,y\in R}(f^{-1}(B)(y)\propto f^{-1}(B)(xy))$$

$$=\prod_{x,y\in R}(B(f(y))\propto B(f(xy)))$$

$$=\prod_{x,y\in R}(B(f(y))\propto B(f(x)f(y)))$$

$$=\prod_{z,w\in H}(B(w)\propto B(zw))$$

$$=[I_L(B)]$$

where $f(x)=z, f(y)=w$.

Similarly, $[I_R(f^{-1}(B))]=[I_R(B)]$.

Therefore $[r(f^{-1}(B))]=[r(B)]$, and $[I(f^{-1}(B))]=[I(B)]$.

Theorem 4 Let $A\in\mathscr{F}_L(R)$, $o\in H$, and let $f:R\to H$ be a surjective homomorphism . Then

$$\vDash (f^{-1}(o)\equiv A)\to I(A).$$

Proof: $[f^{-1}(o)\equiv A]=\prod_{x\in R}(f^{-1}(o)(x)\ \beta\ A(x))$

$$=\prod_{x\in f^{-1}(o)}(1\propto A(x))\cdot\prod_{x\notin f^{-1}(o)}(A(x)\propto o)$$

$$=\prod_{\substack{x,y\in R\\(x+y)\in f^{-1}(o)}}(1\propto A(x+y))\cdot\prod_{x\notin f^{-1}(o)}(A(x)\propto o)$$

$$\leqslant\prod_{(x+y)\in f^{-1}(o)}((A(x)\cdot A(y)\propto A(x+y))\cdot\prod_{x\notin f^{-1}(o)}(A(x)\cdot A(y)\propto o)$$

$$\leqslant\prod_{(x+y)\in f^{-1}(o)}((A(x)\cdot A(y)\propto A(x+y))\cdot\prod_{x\notin f^{-1}(o)}(A(x)\cdot A(y))\propto A(x+y))$$

$$=\prod_{x,y\in R}((A(x)\cdot A(y))\propto A(x+y))$$

$$=[r_1(A)]$$

$$[f^{-1}(o)\equiv A]=\prod_{x\in f^{-1}(o)}(1\propto A(x))\cdot\prod_{x\notin f^{-1}(o)}(A(x)\propto o)$$

$$\leqslant\prod_{x\in f^{-1}(o)}(A(-x)\propto A(x))\cdot\prod_{x\notin f^{-1}(o)}(A(x)\propto A(-x))$$

$$=\prod_{x\in R}(A(x)\propto A(-x))$$

$$=[r_2(A)]$$

Similarly $[f^{-1}(o)\equiv A]\leqslant[r_3(A)]$

$$[f^{-1}(o)\equiv A]=\prod_{x\in f^{-1}(o)}(1\propto A(x))\cdot\prod_{x\notin f^{-1}(o)}(A(x)\propto o)$$

$$\leqslant\prod_{\substack{x,y\in R\\ xy\in f^{-1}(o)}}(1\propto A(xy))\cdot\prod_{\substack{x,y\in R\\ xy\notin f^{-1}(o)}}(A(x)\propto A(xy))$$

$$\leqslant\prod_{\substack{x,y\in R\\ xy\in f^{-1}(o)}}(A(x)\propto A(xy))\cdot\prod_{\substack{x,y\in R\\ xy\notin f^{-1}(o)}}(A(x)\propto A(xy))$$

$$=\prod_{x,y\in R}(A(x)\propto A(xy))$$

$$=[I_L(A)]$$

Similarly, $[f^{-1}(o)\equiv A]\leqslant[I_R(A)]$.

Therefore, we have $[f^{-1}(o)\equiv A]\leqslant[I(A)]$.

Definition 3 Let $A\in\mathcal{F}_L(R)$, $o\in H$, and let $f:R\to H$ be a homomorphism. Then $f_A^{-1}(o)\in\mathcal{F}_L(R)$ is called the L-fuzzifying kernel of the homomorphis f relative to A, if

$$f_A^{-1}(o)=\begin{cases}A(x), & x\in f^{-1}(o),\\ 0, & otherwise,\end{cases}\qquad \text{for any } x\in R.$$

Theorem 5 Let $f:R\to H$ be a homomorphism. Then for any $A\in\mathcal{F}_L(R)$, we have

$$\vDash I(A)\to I(f_A^{-1}(o)).$$

Proof: $[r_1(f_A^{-1}(o))]=\prod_{x,y\in R}((f_A^{-1}(o)(x)\cdot f_A^{-1}(0)(y))\propto f_A^{-1}(o)(x+y))$

$$=\prod_{x,y\in f^{-1}(o)}(A(x)\cdot A(y)\propto A(x+y))$$

$$\geqslant[r_1(A)]$$

$$[r_2(f_A^{-1}(o))]=\prod_{x\in R}((f_A^{-1}(o)(x)\propto f_A^{-1}(o)(-x))=\prod_{x\in f^{-1}(0)}(A(x)\propto A(-x))\geqslant[r_2(A)]$$

Similarly $[r_3(f_A^{-1}(o)]\geqslant[r_3(A)]$,

$$[I_L(f_A^{-1}(o))]=\prod_{x,y\in R}(f_A^{-1}(o)(y)\propto f_A^{-1}(o)(xy))=\prod_{x,y\in f^{-1}(o)}(A(y)\propto A(xy))\geqslant[I_L(A)]$$

Similarly $[I_R(f_A^{-1}(o))]\geqslant[I_R(A)]$

Therefore we have $[I(f_A^{-1}(o))]\geqslant[I(A)]$

Definition 4 Let I be a ideal of R, $A\in\mathcal{F}_L(R)$, and let $N\in\mathcal{F}_L(I)$. Then $A/N\in\mathcal{F}_L(R/I)$ is called the L-fuzzifying factor set of R relative to the ideal I, if

$$x+I\in A/N:=(\exists a)((a\in x+N)\to(a\in A)\wedge(x\in a+N))$$

Theorem 6 (Fundamental theorem of homomorphis) Let $f: R \to H$ be a surjective homomorphism, $A \in \mathcal{F}_L(R)$ and $o' \in H$. Then

$$\vDash r_1(A) \wedge r_2(A) \to (\exists g)((g \in H^{R/f^{-1}(o')}) \to (g(A/f_A^{-1}(o') \equiv f(A)))$$

Proof For any $x + f^{-1}(o') \in R/f^{-1}(o')$, we set $g(x + f^{-1}(o')) = f(x) = y$, Then $[g(A/f_A^{-1}(o')) \equiv f(A)]$

$$= \prod_{y \in H} \Big(\sum_{g(x+f^{-1}(o'))=f(x)=y} (A/f_A^{-1}(o'))(x + f^{-1}(o')) \ \beta \ \sum_{f(x)=y} A(x)\Big)$$

$$\geqslant \prod_{y \in H} \prod_{f(x)=y} \Big(\sum_{a \in R} A(a) \cdot (f_A^{-1}(o')(-a+x)) \ \beta \ A(x)\Big)$$

$$= \prod_{y \in H} \prod_{f(x)=y} \Big(\sum_{f(a)=f(x)} A(a) \cdot A(-a+x) \ \beta \ A(x)\Big)$$

$$= \prod_{y \in H} \prod_{f(x)=y} \Big(\Big(\sum_{f(a)=f(x)} A(a) \cdot A(-a+x) \propto A(x)\Big) \cdot \Big(A(x) \propto \sum_{f(a)=f(x)} A(a) \cdot A(-a+x)\Big)\Big)$$

$$\geqslant \prod_{y \in H} \prod_{f(x)=y} \Big(\prod_{f(a)=f(x)} (A(a) \cdot A(-a+x) \propto A(x)) \cdot (A(x) \propto (A(x) \cdot A(o)))\Big)$$

$$\geqslant \prod_{y \in H} \prod_{f(x)=y} \Big(\prod_{f(a)=f(x)} (A(a) \cdot A(-a+x) \propto ([r_1(A)] \otimes (A(a) \cdot A(-a+x)))) \cdot (A(x) \propto A(0))\Big)$$

$$\geqslant \prod_{y \in H} \prod_{f(x)=y} [r_1(A)] \cdot [r_4(A)]$$

$$= [r_1(A) \wedge r_4(A)]$$

Where, from **Theorem 1(2)**, we use $A(x) \propto A(o) \geqslant [r_4(A)]$

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