On the Choquet integral of non – monotonic fuzzy measures*

Wang Guijun[a], He Walin[b]
[a] Department of Mathematics, Tonghua Teachers' College, Tonghua, Jilin, 134002, P.R. China.
[b] Tonghua 10th Middle School, Tonghua, Jilin, 134001, P.R. China.

Abstract In this paper, on the basis of the concept of non – monotonic fuzzy measures introduced by T. Murofushi and M. Sugeno [4], We study further some special properties with respect to this kind of Choquet integral, and being directed against this kind of non – monotonic fuzzy measures added to continuity, we give monotonic bounded convergence theorem of this kind of new Choquet integral.

Keywords Non – monotonic fuzzy measures; non – monotonic continuous fuzzy measures; measurable functions; μ – integrable; Choquet integrals.

1 Introduction

Since G. Choquet [1] presented Choquet integrals in 1953, being directed against different conditions and in the light of different ways, Choquet integrals were studied widely and deeply by many authors, e.g. [2,3,4,5,6,7] etc. The non – monotonic fuzzy measures which are set functions without monotonicity and continuity were introduced at the first time in [4], and the Choquet integral were being defined from this. With the help of the concepts of bounded variations and total variations, by the ways in functional analysis, a necessary and sufficient conditions which an ordinary real valued functional may be represented by this kind of Choquet integral was given on the general bounded measurable functions spaces. But the fuzzy measures referred to above have not monotonicity and continuity, therefore, corresponding Choquet integral has not monotonicity and continuity too. Thus, the convergence of this kind of Choquet integral can not be studied further. Being directed against the fuzzy valued functions taken valued fuzzy numbers, fuzzy valued functionals (i.e., fuzzy Choquet integrals) in the sense of Sugeno's fuzzy measures were defined in [6,

* This work has been supported by the Youth Science Foundation of Jilin education commission. P.R. China

7], and their general properties and convergence theorem were studied.

The main purpose of this paper is to re – define a new Choquet integral, on non – negative bounded measurable functions spaces by defining non – monotonic fuzzy measures (continuous). Consequently, we obtain their monotonic bounded convergence theorem being similar to classicall Lebesgue integrals.

The Choquet integral defined in this paper has not still monotonicity, thus, some results, such as Levi's theorem, Fatou's lemma and dominated convergence theorem etc., remain to be discussed further in the future.

2 Non – monotonic fuzzy measure and the Choquet integral.

Let X be an arbitrary fixed set, $\mathscr{A}$ be a σ – algebra formed by the subsets of X, and $(X, \mathscr{A})$ be a measurable space, R^+ denote the interval $[0, +\infty)$, *i.e.*, $R^+ = [0, +\infty)$

Definition 2.1 Let $(X, \mathscr{A})$ be an arbitrary measurable space, a set function $\mu: \mathscr{A} \to [0,1]$, and

(1) $\mu(\phi) = 0$, $\mu(X) = 1$

(2) If $\{A_n\}$ is an arbitrary monotonic sequence of sets in $\mathscr{A}$, then $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$.

If set functions μ only satisfies (1), then μ is called a non – monotonic fuzzy measure; If set functions μ satisfies both (1) and (2), then μ is called a non – monotonic continuous fuzzy measure. At this time, corresponding $(X, \mathscr{A}, \mu)$ is called a non – monotonic fuzzy measure space and a non – monotonic continuous fuzzy measure space respectively.

Definition 2.2 Let $(X, \mathscr{A})$ be an arbitrary measurable space, f be a real valued function on X, Then f is called a measurable function on $(X, \mathscr{A})$, if $f_\lambda \triangleq \{x \in X \mid f(x) \geq \lambda\} \in \mathscr{A}$ for all $\lambda \in R^+$.

Let $BM(X, \mathscr{A}) = \{f \mid f$ is a real – valued bounded measurable function on $(X, \mathscr{A})\}$

Defining the norm: $\|f\| = \sup_{x \in X} |f(x)|$ on the space $BM(X, \mathscr{A})$, then we are easy to be proved that $BM(X, \mathscr{A})$ constitutes a Banach space, please refer to paper[3].

In this paper, for convenience sake, we only discuss problems on non-negative bounded measurable functions spaces.

Let $BM^+(X, \mathscr{A}) = \{f \in BM(X, \mathscr{A}) \mid f(x) \geqslant 0, \text{ for any } x \in X\}$.

Definition 2.3 Let $(X, \mathscr{A}, \mu)$ be a non-monotonic fuzzy measure space, $f \in BM^+(X, \mathscr{A})$, let

$$(C)\int f d\mu \triangleq \int_0^{+\infty} \mu(f_\lambda)\, d\lambda .$$

Where $f_\lambda = \{x \mid f(x) \geqslant \lambda\}$, $\lambda \in R^+$, the right integral is a integral in the sense of Lebesgue.

If above integral $\int_0^{+\infty} \mu(f_\lambda)\, d\lambda$ exists and its integral-valued is finite, then we call f μ-integrable, at the monent, $(C)\int f d\mu$ is called a Choquet integral of f with respect to μ, it is simply called the Choquet integral.

Note 1 Obviously, $(C)\int f d\mu$ has non-negativeity for any $f \in BM^+(X, \mathscr{A})$; But μ has not monotonicity, even if $f, g \in BM^+(X, \mathscr{A})$ and $f(x) \leqslant g(x)$ for all $x \in X$, although $f_\lambda \subset g_\lambda$, for any $\lambda \in R^+$, then $\mu(f_\lambda) \leqslant \mu(g_\lambda)$ may not hold. Hence, the corresponding Choquet integral $(C)\int f d\mu$ has not monotonicity. This brings many difficulty for us to study further the convergence of this kind of integral.

In the following discussion, We will give some properties being not referred to in [4].

Theorem 2.1 Let μ_1 and μ_2 be two non-monotonic fuzzy measures on measurable space $(X, \mathbf{A})$,
$f \in BM^+(X, \mathscr{A})$ and f be μ_i-integrable, $i = 1,2$. If for all $A \in \mathscr{A}$, $\mu_1(A) \leqslant \mu_2(A)$. Then $(C)\int f d\mu_1 \leqslant (C)\int f d\mu_2$.

Proof By the monotonicity of Lebesgue integrals, it is obvious.

Theorem 2.2 If f is a constant valued function, i.e., $f(x) \equiv a \geqslant 0$ for every $x \in X$. Then with respect to an arbitrary non-monotonic fuzzy measure μ, f is μ-integrable, and $(C)\int f d\mu = a$.

Proof For any $\lambda \in R^+$.

whenever $\lambda > a$, $\mu(f_\lambda) = \mu(\{x \mid f(x) \geqslant \lambda\}) = \mu(\phi) = 0$

whenever $\lambda \leqslant a$, $\mu(f_\lambda) = \mu(\{x \mid f(x) \geqslant \lambda\}) = \mu(X) = 1$

Thus,

$$\begin{aligned}(C)\int f d\mu &= \int_0^{+\infty} \mu(f_\lambda)\, d\lambda \\ &= \int_{[0,a]} \mu(f_\lambda)\, d\lambda + \int_{(a,+\infty)} \mu(f_\lambda)\, d\lambda \\ &= \int_{[0,a]} \mu(X)\, d\lambda + \int_{(a,+\infty)} \mu(\phi)\, d\lambda \\ &= \int_{[0,a]} d\lambda = a < +\infty\end{aligned}$$

Therefore, f is μ-integrable and $(C)\int f d\mu = a$

Theorem 2.3 Let $(X, \mathscr{A}, \mu)$ be a non-monotonic fuzzy measure space, $f \in BM^+(X, \mathscr{A})$, and f be μ-integrable, $a \geqslant 0, b > 0$. Then $a + bf$ is μ-integrable too, and $(C)\int (a + bf)\, d\mu = a + b(C)\int f d\mu$.

Proof

$$\begin{aligned}(C)\int (a + bf)\, d\mu &= \int_0^{+\infty} \mu(\{x \mid a + bf(x) \geqslant \lambda\})\, d\lambda \\ &= \int_{[0,a]} \mu(\{x \mid f(x) \geqslant \frac{\lambda - a}{b}\})\, d\lambda \\ &\quad + \int_{(a,+\infty)} \mu(\{x \mid f(x) \geqslant \frac{\lambda - a}{b}\})\, d\lambda \\ &= I_1 + I_2\end{aligned}$$

where $I_1 = \int_{[0,a]} \mu(X)\, d\lambda = \int_{[0,a]} d\lambda = a$

For I_2, Let $\eta = \frac{\lambda - a}{b}$, *then* $d\lambda = bd\eta$

Consequently, we have

$$I_2 = \int_{(a,+\infty)} \mu(\{x \mid f(x) \geqslant \frac{\lambda - a}{b}\})d\lambda$$
$$= \int_0^{+\infty} \mu(\{x \mid f(x) \geqslant \eta\})bd\eta$$
$$= b(C)\int fd\mu < +\infty$$

Therefore

$(C)\int (a + bf)d\mu = a + b(C)\int fd\mu$ and $a + bf$ is μ - integrable.

Note 2 Evidently, this kind of Choquet integral satisfy positive homogeneity with respect to the non - negative bounded measurable functions, i.e, Whenever $a = 0$, $(C)\int bfd\mu = b(C)\int fd\mu\,(b > 0)$. But they do not satisfy additivity. And so, this kind of Choquet integral is non - linear.

Theorem 2.4 Let $(X, \mathscr{A}, \mu)$ be a non - monotonic fuzzy measure space, $A \in \mathscr{A}$, χ_A be a characteristic function on A. i.e., $\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$. Then χ_A is μ - integrable and $(C)\int fd\mu = \mu(A)$.

Proof $(C)\int \chi_A d\mu = \int_{[0,1]} \mu(\{x \mid \chi_A(x) \geqslant \lambda\})d\lambda + \int_{(1,+\infty)} \mu(\{x \mid \chi_A(x) \geqslant \lambda\})d\lambda$

$$= \int_{[0,1]} \mu(A)d\lambda + \int_{(1,+\infty)} \mu(\phi)d\lambda$$
$$= \mu(A)$$

Corollary 2.5 If f is a non - negative simple function on the non - monotonic fuzzy measure space $(X, \mathscr{A}, \mu)$. i.e., $f(x) = \sum_{i=1}^{n} a_i \chi_{D_i}(x)$, where $0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, $X = \bigcup_{i=1}^{n} D_i$ and $D_i \cap D_j = \phi$, $i \neq j$. Then $(C)\int fd\mu =$

$\sum_{i=1}^{n}(a_i - a_{i-1})\mu(A_i)$, where $A_i = \bigcup_{k=i}^{n} D_k, i = 1,2\cdots,n, a_0 = 0$

3 Monotonic bounded convergence theorem of the Choquet integral

In order to discuss the bounded convergence theorem of this kind of Choquet integral, in this section, we only study problems on the non – monotonic continuous fuzzy measures spaces.

Definition 3.1 Let $(X,\mathscr{A},\mu)$ be a non – monotonic continuous fuzzy measure space, E be an arbitrary bounded set on R^+, $f \in BM^+(X,\mathscr{A})$. Re – define the Choquet integral: $(\mathrm{C})\int_E f d\mu \triangleq \int_E \mu(f_\lambda) d\lambda$

Where $f_\lambda = \{x \mid f(x) \geqslant \lambda\}, \lambda \in E$, the right integral is still classical Lebesgue integral.

From this definition, if the function f is a μ – integrable function on $BM^+(X,\mathscr{A})$, then we are easy to get that.

$$(C)\int f d\mu = \int_E \mu(f_\lambda) d\lambda \leqslant \int_0^{+\infty} \mu(f_\lambda) d\lambda < +\infty$$

Thus, the above Chquet integral defined once again is always existent. Next, We will give an important result of this paper, i.e.,monotonic bounded convergence theorem.

Theorem 3.1 (Monotonic bounded convergence theorem) Let $(X,\mathscr{A},\mu)$ be a non – monotonic continuous fuzzy measure space, $f_n: X \to R^+$ be a monotonic sequence of μ – integrable functions which converge to $f: X \to R^+$, E be a bounded set on R^+, and f be μ – integrable, too. Then $\lim_{n\to\infty}(C)\int_E f_n d\mu = (C)\int_E f d\mu$

Proof Without loss of generality, Let $\{f_n\}$ be a monotonic decreasing sequence of functions, for every $\lambda \in K \subset R^+$, let

$$(f_n)_\lambda \triangleq \{x \mid f_n(x) \geq \lambda\}, n = 1,2,\cdots$$

As $\{f_n\}$ and f are μ – integrable, Consequently, they are measurable functions on $BM^+(X,\mathscr{A})$.

And so, for any natural number n, $(f_n)_\lambda \in \mathscr{A}$, $f_\lambda \in A$, and $\{(f_n)_\lambda\}$ is a monotonic decreasing sequence of sets on $\mathscr{A}$ with respece to natural number n. i.e.,

$$(f_{n+1})_\lambda \subset (f_n)_\lambda, n = 1,2\cdots$$

Therefore, we have

$$\lim_{n\to\infty}(f_n)_\lambda = \bigcap_{n=1}^{\infty}(f_n)_\lambda = f_\lambda, \text{ for every } \lambda \in E \subset R^+$$

Actually, on the one hand, if $x \in \bigcap_{n=1}^{\infty}(f_n)_\lambda$, then $x \in (f_n)_\lambda$ or $f_n(x) \geq \lambda$ for arbitrary natural number n.

By the hypothesis $\lim_{n\to\infty} f_n(x) = f(x)$, for all $x \in X$, and according to the property of inequality of limit, we know that $f(x) = \lim_{n\to\infty} f_n(x) \geq \lambda$,

Hence $x \in f_\lambda$, it follows that $\bigcap_{n\to\infty}^{\infty}(f_n)_\lambda \subset f_\lambda$

On the other hand, if $x \notin \bigcap_{n=1}^{\infty}(f_n)_\lambda$, then there exists a natural number n_0 at least such that $x \notin (f_{n_0})_\lambda$, i.e., $f_{n_0}(x) < \lambda$. By $\{f_n\}$ is decreasing, we can derive

$$f(x) \leq f_{n_0}(x) < \lambda$$

Hence $x \notin f_\lambda$, and $f_\lambda \subset \bigcap_{n=1}^{\infty}(f_n)_\lambda$

Thereby, We obtain $\bigcap_{n=1}^{\infty}(f_n)_\lambda = f_\lambda$, for every $\lambda \in E$

Since the non – monotonic fuzzy measure μ has continuity, We can infer that

$$\lim_{n\to\infty}\mu((f_n)_\lambda) = \mu(\lim_{n\to\infty}(f_n)_\lambda) = \mu(f_\lambda)$$

In accordance with the bounded property of μ, $0 \leq \mu((f_n)_\lambda) \leq 1$ is clear.

Taking advantage of the bounded convergence theorem of classical Lebesgue integrals, we get

$$\lim_{n\to\infty}\int_E \mu((f_n)_\lambda)\,d\lambda = \int_E \lim_{n\to\infty}\mu((f_n)_\lambda)\,d\lambda = \int_E \mu(f_\lambda)\,d\lambda$$

Consequently $$\lim_{n\to\infty}(C)\int_E f_n d\mu = (C)\int_E f d\mu$$

Corollary 3.2 Let $(X, \mathbf{A}, \mu)$ be a non – monotonic continuous fuzzy measure space, $\{f_n\}$ be a monotonic sequence of μ – integrable functions converging to f a.e. on $BM^+(X, \mathscr{A})$, and $f: X \to R^+$ be μ – integrable, too. Then $\lim_{n\to\infty}(C)\int_E f_n d\mu = (C)\int_E f d\mu$

4. Conclusion

In this paper, We only obtain monotonic bounded convergence theorem of the Choquct integral on the non – monotonic continuous fuzzy measures spaces. But the others results, such as Levi ' s Theorem, Fatou ' s Lemma and Lebesgue's Dominated Convergence Theorem etc. can not be discussed for the present. The reason is because this kind of Choquet integral themself have not monotonicity, And so, this is a new topic remaining to be studied in the future.

References

〔1〕 G. Choquet. Theory of capacities. Ann. Inst. Fourier 5(1953) 131 – 295.

〔2〕 T. Murofushi and M. Sugeno. An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure. Fuzzy Sets and Systems 29(1989) 201 – 227.

〔3〕 T. Murofushi and M. Sugeno. A theory of fuzzy measures: representations. the Choquet integral and null sets. J. Math. Anal. Appl. 159 (1991) 532 – 549.

〔4〕 T. Murofushi, M. Sugeno and M. Machida. Non – monotonic fuzzy measures and the Choquet integral. Fuzzy Sets and Systems 64(1994) 73 - 86.

〔5〕 L.C. Jang, B.M.Kil, Y.K.Kim and J.S.Kwon. Some properties of Choquet integrals of set - valued functions. Fuzzy Sets and Systems 91 (1997) 95 - 98.

〔6〕 Wang Guijun and Li Xiaoping. Fuzzy Choquet integrals and their limit theorem. J.Lanzhou University, 32(1996) 311 - 315.(in Chinese)

〔7〕 Wang Guijun and Li Xiaoping. On the convergence of the fuzzy valued functional. J. Fuzzy Math. 5(2)(1997) 431 - 438.

〔8〕 Wang Guijun. The interval - valued fuzzy measure. J.Fuzzy Math. 6(3) (1998) 525 - 532.

〔9〕 Wang Guijun and Li Xiaoping. On the weak convergence of fuzzy measures and metric of fuzzy measures. Fuzzy Sets and Systems.(to appear)