The theory of generalized fuzzy integrals

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Abstract: In this paper, the theory of generalized fuzzy integral will be summarized. It includes single-valued generalized fuzzy integrals, set-valued generalized fuzzy integrals and fuzzy valued generalized fuzzy integrals.
Keywords: Fuzzy integral; Fuzzy measure; single-valued function; set-valued function; fuzzy-valued function.

1. Introduction

Since Sugeno[8] brought out the concepts of fuzzy measures and fuzzy integrals, the theory had been made much deeper by Ralescu[6], Wang[9], Zhao[21] and someothers. As an extension of Sugeno's fuzzy integral and Zhao's (N) fuzzy integral, we have put forward the generalized fuzzy integral in [10], then given it detail discussions in [2],[11],[12],[13], etc. Basing on these, the theory of generalized fuzzy integrals of set-valued functions is established in [17]. Furtherly, by introducing the concept of fuzzy number fuzzy measure, the theory of generalized fuzzy integrals of fuzzy-valued function is built up in [3],[4] The paper's purpose is to summarize the main concepts and results on this theory, and the discussion route is from the case of single-value, to the case of set-valued and fuzzy-value.

Through out the paper, R^+ denotes $[0, \infty]$, X is an abstract nonempty set, $\mathscr{A}$ is a σ-algebra formed by the subsets of X, $(X, \mathscr{A})$ is the measurable space, set-function $\mu : \mathscr{A} \to [0, \infty]$ is a fuzzy measure under Sugeno's sense, $(X, \mathscr{A}, \mu)$ is the fuzzy measure space, $F(X)$ denotes the set of all nonnegative $\mathscr{A}$-measurable functions. $M(X)$ is the set of all fuzzy measures on $(X, \mathscr{A})$.

The remainder of the paper consists of three sections. In section 2, we will show single-valued generalized fuzzy integrals; in section 3, we will give set-valued generalized fuzzy integrals, and in section 4, we will discuss fuzzy-valued generalized fuzzy integrals.

2. Singele-valued generalized fuzzy integrals

In this section, we first define a generalized triangle norm, then define the generalized fuzzy integral, finally show its some basic propositions and convergence theorems.

Definition 2.1[10] *Let* $D = [0, \infty] \times [0, \infty] \setminus \{(0, \infty), (\infty, 0)\}$, *A mapping* $S : D \to$

$[0,\infty]$ *is said to be a generalized triangle norm if it satisfies the following conditions*:

(i) $S(0,x)=0$, *for each* $x\in(0,\infty)$, *and there exists* $e\in(0,\infty]$, *s.t.* $S(x.e)=x$ *for each* $x\in(0,\infty]$, *e is called identity of* S;

(ii) $a\leqslant b$, $c\leqslant d$ *implies* $S(a,c)\leqslant S(b,d)$;

(iii) $S(a,b)=S(b,a)$;

(iv) $\{(x_n,y_n)\}\subset D,(x,y)\in D$, *if* $x_n\uparrow x$, $y_n\downarrow y$, *then* $S(x_n,y_n)$ *converges to* $S(x,y)$.

Definition 2.2[10] *Let* $f\in F(X)$, $A\in\mathscr{A}$, $\mu\in M(X)$. *Then the generalized fuzzy integral of f over A with respect to* μ *is as follows*:

$$\int_A f d\mu=\bigvee_{\alpha\geqslant 0}S(\alpha,\mu(A\cap F_\alpha))$$

where $F_\alpha=\{x\in X:f(x)\geqslant\alpha\}$

Theorem 2.1[10] *Generalized fuzzy integrals have following properties*:

(i) $f_1\leqslant f_2$ *implies* $\int_A f_1 d\mu\leqslant\int_A f_2 d\mu$;

(ii) $A\subset B$ *implies* $\int_A f d\mu\leqslant\int_B f d\mu$;

(iii) $\mu(A)=0$ *implies* $\int_A f d\mu=0$;

(iv) $\int_A f d\mu=\int_X X_A\cdot f d\mu$, *where* X_A *is the characteristic function of* A;

(v) $\int_A c d\mu=S(c,\mu(A))$;

(vi) $\int_A(c\vee f)d\mu=\int_A c d\mu\vee\int_A f d\mu$;

(vii) $\int_A f d\mu=\int_0^\infty\mu(A\cap F_\alpha)d\alpha$, *where* α *is the Lebesgue measure, the rihgt-hand integral is also generalized fuzzy integral*.

Theorem2.2[2] (*Generalized monotone convergence theorem*) *Let* $\{f_n(n\geqslant 1),f\}\subset F(X),\{\mu_n(n\geqslant 1),\mu\}\subset M(X)$.

(i) *If* $f_n\uparrow f$, $\mu_n\uparrow\mu$, *then* $\int_A f_n d\mu_n\uparrow\int_A f d\mu$;

(ii) *If* $S(\frac{1}{n},\infty)\rightarrow 0(n\rightarrow\infty)$, *and for* $\forall\varepsilon>0,\exists\delta\in(0,\varepsilon),n_0\in N$, *s.t.* $\mu(f_{n0}\geqslant c_0+\delta)\cap A)<\infty$, *where* $c_0=\sup\{a>0:S(a,\infty)\leqslant\int_A f d\mu\}$, *then* $f_n\downarrow f,\mu_n\downarrow\mu$ *implies* $\int_A f_n d\mu_n\downarrow\int_A f d\mu$.

Remark. There are some other discussions on single-valued generalized fuzzy integral, for instance, various kinds of convergence theorems in [12], Riesz represantation theorem in [15], generalized fuzzy integral on L-fuzzy sets in [13], fuzzy measure defined by generalized fuzzy integral, weak convergence of fuzzy measure under generalized fuzzy integrals, and level convergence theorem in [20], etc. At last, it should be pointed out, a kind of more general fuzzy integral——generalized seminormed fuzzy integral is also built up in [20], it can make generalized fuzzy integral and seminormed fuzzy integral[7] as special.

3. Set-valued generalized fuzzy integrals

Let $P(R^+)$ be the power set of R^+. A set-valued function is a mapping $F:X\rightarrow P(R^+)\backslash\{\varnothing\}$. The concepts of measurability, convexity, closedness and others on set-valued functions can be found in [1]. Integrals of set-valued functions were in [1]. Here, as an extension of set-valued fuzzy integral[16], we will brought out the theory of set-valued generalized fuzzy integrals.

Definition 3.1[17] *Let F be a set-valued function*, $\mu\in M(X)$, $A\in\mathscr{A}$. *The generalized fuzzy integral of F over A with respect to* μ *is as follows*:

$$\int_A F d\mu=\{\int_A f d\mu: f\in S(F)\}$$

where $S(F)$ *is the set of all measurable selections of* F, "$\int_A$" *is the generalized fuzzy integral.*

Theorem 3.1[17] *Let* μ *be an m-continuous fuzzy measure. Then set-valued generalized fuzzy integral has following properties:*

(i) F *is closed-valued implies* $\int_A F d\mu$ *is closed;*

(ii) F *is convex-valued implies* $\int_A F d\mu$ *is convex;*

(iii) $\int_A coF d\mu = co\int_A F d\mu$.

Theorem 3.2[17] *Let* $F_n (n \geqslant 1)$ *be set-valued functions,* $\mu(X) < \infty$. *Then*

(i) $\text{Limsup} \int_A F_n d\mu \subset \int_A \text{Limsup} F_n d\mu$,

(ii) μ *is null-additive and m-continuous implies*

$$\int_A \text{Liminf} F_n d\mu \subset \text{Liminf} \int_A F_n d\mu.$$

Remark This section were adopted from [17], for more details, refer to [16,17].

4. Fuzzy – valued generalized fuzzy integrals.

A fuzzy number $\tilde{r}$ is a normal, closed-convex fuzzy set on R^+. We use $\tilde{R}^+$ to denote the set of all fuzzy number. On $\tilde{R}^+$, the Hausdorff metric is as $D(\tilde{r}, \tilde{p}) = \sup_{\alpha \geqslant 0}((r_\lambda^- - p_\lambda^-) \vee (r_\lambda^+ - p_\lambda^+))$, where $r_\lambda^{-(+)} = \inf(\sup)\{r : \tilde{r}(r) \geqslant \lambda\}$. A fuzzy-valued function is a mapping $\tilde{f}: X \to \tilde{R}^+$, it is said to be measurable, if its λ-level interval-valued function $\bar{f}_\lambda : X \to I(R^+)$ is measurable for every $\lambda \in (0,1]$. A fuzzy number fuzzy measure is a mapping $\tilde{\mu}: \mathscr{A} \to \tilde{R}^+$, satisfies normality from below, monotonicity, continuity. In this section, we will show the concept of generalized fuzzy integrals of fuzzy-valued functions with respect to fuzzy number fuzzy measures.

Definition 4.1[14] *Let* $\tilde{f}$ *be a fuzzy-valued function,* $\tilde{\mu}$ *be a fuzzy number fuzzy measure,* $A \in \mathscr{A}$. *Then the generalized fuzzy integral of* $\tilde{f}$ *over* A *with respect to* $\tilde{\mu}$ *is as follows:*

$$(\int_A \tilde{f} d\tilde{\mu})(r) = \sup\{\lambda \in (0,1] : r \in \int_A \bar{f}_\lambda d\bar{\mu}_\lambda\}$$

Theorem 4.1[4] *Let* $\tilde{f}$ *be a fuzzy-valued function,* $\tilde{\mu}$ *be a fuzzy number fuzzy measure,* $A \in \mathscr{A}$. *Then* $\int_A \tilde{f} d\tilde{\mu} \in \tilde{R}^+$, *and* $(\int_A \tilde{f} d\tilde{\mu})_\lambda = \int_A \bar{f}_\lambda d\bar{\mu}_\lambda$, *where* $\lambda \in (0,1]$.

Remark. By using the above theorem and results of single – valued generalized fuzzy integrals, we can easily obtain various kinds of properties and convergence theorems about the fuzzy-valued generalized fuzzy integral. For the details, refer to [3,4]. For the further work, refer to generalized fuzzy integrals on fuzzy sets[5][18][19].

Concluding remark

Up to here, we have summarized the theory of generalized fuzzy integral of three kinds of functions, they are single-valued functions, set-valued functions and fuzzy-valued functions. For the application of single-valued generalized fuzzy integral, it has been partly shown in [11]. As to the others applications, it should be furtherly investigated.

At last, we pointed out as an extension of single-valued generalized fuzzy integral, lattice-valued generalized fuzzy integral has been studied in detail[20], it is the most general one as far as we know.

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