Some equivalent depictions of fuzzy codes

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Abstract

In this paper, the algebraic properties of fuzzy codes have been discussed, and several equivalent depictions of fuzzy codes, fuzzy prefix (suffix, biprefix) codes have been given.

1.Preliminaries

The notion of the fuzzy language was first introduced by zaeleh in 1969[1]. Since then, in the study of fuzzy formal language, a lot of excellent results have been achiedve by researches. Particiularly, the study of fuzzy formal language extended the applicable area of fuzzy set theory and reduced the difference between formal language and natural language[2]. J.Z.Shen introduced the con cepts of fuzzy base on fuzzy monoid in[4]. Based on[5], Shen in troduced the concepts of fuzzy code, fuzzy prefix code and maximal fuzzy prefix code. In this paper, the algebraic properties of fuzzy codes have been discussed, and several equivalent depiction of fuzzy codes, fuzzy prefix (Suffix, biprefix) codes have been given.

In the following text, we suppose that X (Y, Z) is an alphabet with $1\leqslant|X|$ (|Y|,|Z|)< ∞ and X^* (X^+) ($Y^*(Y^+)$,$Z^*(Z^+)$) is the free monoid (semigroup)generated from X (Y,Z) with the operation of adjoin. F stand for "fuzzy" and F(X) denotes the set of all fuzzy subsets of X, $A\in F(X^*)$ is called F-language on the free monoid X^*.e is the identity of X^*.

Definition 1.1. Let $A,B\in F(X^*)$, for any $x\in X^*$, $(A-B)(x)=\begin{cases} A(x), \text{if } B(x)=0 \\ 0 \quad, \text{if } B(x)>0 \end{cases}$

$(AB)(x)=\sup\limits_{yz=x}\min(A(y),B(z))$.

Definition 1.2. Let S be a Semigroup, $B\in F(S)$ is called a F-subsemigroup of S if $B(xy)\geqslant\min(B(x),B(y))$ for any $x,y\in S$.

Definition1.3[4]. A F-subesemigroup A of X^* is called a F-submonoid of X^* if A(e)=1

Definition1-4[4]. Let $A\in F(X^*)$ be a F-submonoid of X^*, $B\subseteq A$ is called a F-based of A if B(e)=0 and

(B_1) for any $x\in$ suppA-{e}, $B^+(x)\geqslant A(x)$;

(B_2) for any $x\in$ suppA-{e},$x_i,y_i\in X^*$,i=1,......,n;j=1,......,m,

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and $x=x_1x_2\ldots\ldots x_n=y_1y_2\ldots\ldots y_m$, inply that $\min(B(x_1),\ldots\ldots,B(x_n),B(y_1),\ldots\ldots,B(y_m)) \propto \min([m=n],[x_1=y_1],\ldots\ldots[x_n=y_n]) \geqslant A(x)$, where $a \propto b = 1$ if $a \leqslant b$; $a \propto b = b/a$ if $a>b$; $[m=n]=1$ if $m=n$; $[m=n]=0$ if $m \neq n$.

Definition1.5[4]. A F-submonoid A of X^* is called F-free submonoid if there exists F-based of A such that $B^*=A$.

Definition1.6[5]. $\Phi \neq A \in F(X^+)$ is called a F-code on X if A is a F-based of A^+.

Definition1.7[5]. A nonempty F-language $\Phi \neq A \in F(X^+)$ is a F-prefix code if $A \cap AX^+ = \Phi$.

Definition1.8. Let X be an alphabet, A subset A of the free monoid X^* is a code over X if is for all $n,m \geqslant 1$ and $x_1\ldots\ldots, x_n$, $x'_1,\ldots\ldots, x'_m \in A$, $x_1x_2\cdots\cdots x_n = x'_1x'_2\cdots\cdots x'_m$ implies n=m and $x_i = x'_i$ for $i=1,\cdots\cdots,n$.

Proposeiton1.9. (1) A code never contains the empty word e (2) Any subset of a code is a code. Particularly, the empty set is a code.

Definition1.10. Let A be a subset of X^*, then A is a prefix (suffix) set if no element of A is a proper left (right) factor of another element in A. A is called a biprefix set if it is both prefix and suffix.

Defimtion1.11. A prefix (suffix, biprefix) code is a prefix (suffix,biprefix) set which is a code, that is distinct from {e}.

2.Epuivalent depiction of F-code

Theorem2.1. Let $A \in F(X^*)$ be a F-subset of the free monoid X^*. Then A is a F-code $\Leftrightarrow$ (I) if for any $n,m \geq 1$ and $x_1,\cdots\cdots,x_n$, $x'_1,\cdots\cdots,x'_m \in X^*$, the condition $x_1x_2\cdots\cdots x_n = x'_1x'_2\cdots\cdots x'_m = x$ inplies $\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m)) \propto \min([m=n],[x_1=x'_1],\cdots\cdots,[x_n=x'_n]) \geq A^+(x)$. Where $a \propto b = 1$, if $a \leq b$; $a \propto b = b/a$, if $a > b$ for any $a,b \in [0,1]$;

$$[x=y]=\begin{cases}1, if\ x=y\\ 0, if\ x \neq y\end{cases}.$$

Proof. "$\Leftarrow$" Since e.e=e then $\min(A(e),A(e),A(e)) \propto \min([2=1],[e=e]) \geq A^+(e)$, $A(e) \propto 0 \geq A^+(e)$, so $A(e)=0$. And from (I) we can obtain the following codition easily, for any $x \in \operatorname{supp} A - \{e\}$, $x_i, y_j \in X^*$, $i=1,2,\cdots\cdots,n$, $j=1,2,\cdots\cdots,m$, $x = x_1x_2\cdots\cdots x_n = y_1y_2\cdots\cdots y_m$ implies $\min(A(x_1),\cdots\cdots,A(x_n),A(y_1),\cdots\cdots,A(y_m)) \propto \min([m=n],[x_1=y_1],\cdots\cdots,[x_n=y_n]) \geq A^+(x)$, So A is a F-code.

"$\Rightarrow$" For any $n,m>1$ and $x_1\cdots\cdots x_n, x'_1,\cdots\cdots,x'_m \in X^*$, $x = x_1x_2\cdots\cdots x_n = x'_1\cdots\cdots x'_m$. If $A(x_i)>0$ $i=1\cdots\cdots n$, $A(x'_j)>0$, $j=1,\cdots\cdots,m$, and $x \neq e$ then $A^+(x)>0, x \in \operatorname{supp} A^+ - \{e\}$, by the definition1.6 we have that $\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m)) \propto \min([m=n],[x_1=x'_1],\cdots\cdots,[x_n=x'_n]) \geq A^+(x)$. If $x=e$, obviously $x_i=e, i=1,\cdots\cdots,n$ $x''_j=e, j=1,\cdots\cdots,m$. as $A(e)=0$, So $\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m))=0$, and $\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m)) \propto \min([m=n],[x_1=x'_1],\cdots\cdots,[x_n=x'_n]) \geq A^+(x)$.

If there exists x_{i_0} or x'_{j_0} such that

$A(x_{i_0})=0$, $1\le i_0\le n$ or $A(x'_{i_0})=0$ $1\le j_0\le m$

then

$\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m))=0$,and

$\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m))$ $\propto$

$\min([n=n],[x_1=x'_1],\cdots\cdots,[x_n=x'_n])\ge A^+(x)$.

Propostion2.2 Any subset of a F-code is a F-code. Especially the empty set is a F-code.

Proof. Let A be a F-code, and suppose B is a subset of A, for any $n,m\ge 1$ and $x_1,\cdots\cdots,x_n,x'_1,\cdots\cdots,x'_m\in X^*$, $x_1\cdots\cdots x_n=x'_1\cdots\cdots x'_m=x$, since A is a F-code, we have

$\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m))$

$\propto$ $\min([n=m],[x_1=x'_1],\cdots\cdots,[x_n=x'_n])\ge A^+(x)$.

If $A^+(x)>0$,then

$\min([n=m],[x_1=x'_1],\cdots\cdots,[x_n=x'_n])=1$ or

$\min([n=m],[x_1=x'_1],\cdots\cdots,[x_n=x'_n])$

$=\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m))=0$.

Since $B\subset A$, $B(x_i)\le A(x_i)$, $B(x'_j)\le A(x'_j)$, $i=1,\cdots\cdots,n$; $j=1,\cdots\cdots,m$ .Thus

$\min(B(x_1),\cdots\cdots,B(x_n),B(x'_1),\cdots\cdots,B(x'_m))$ $\propto$

$\min([n=m],[x_1=x'_1],\cdots\cdots,[x_n=x'_n])\ge B^+(x)$. If $A^+(x)=0$, then $B(x)=0$. The result is obviously. By theorenm2.1, B is a F-code.

Theorem2.3. $A\in F(X^*)$ is a F-code

$\Leftrightarrow$ $A_\lambda\subseteq X^*$ is a code for any $\lambda\in(0,1]$.

Proof. "$\Rightarrow$" For any n,m $\geqslant$ 1 and $x_1\cdots\cdots x_n,x'_1,\cdots\cdots,x'_m\in A_\lambda$, $x_1\cdots\cdots x_n=x'_1\cdots\cdots x'_m=x$ Since A is a F-code then

$\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m))$

$\min([n=m],[x_1=x'_1],\cdots\cdots,[x_n=x'_n])\ge A^+(x)$

$\ge A''(x)\ge\lambda>0$, thus

$\min([n=m],[x_1=x'_1],\cdots\cdots,[x_n=x'_n])=1$,

i.e. n=m , $x_i=x_j$ for $i=1,\cdots\cdots,n$, therefore A_λ is a code for $\lambda\in(0,1]$.

"$\Leftarrow$" If A is not a F-code ,then there exists $w\in X^*$, $w=x_1\cdots\cdots x_n=y_1\cdots\cdots y_m$ such that

$\min(A(x_1),\cdots\cdots,A(x_n),A(y_1),\cdots\cdots,A(y_m))$ $\propto$

$\min([n=m],[x_1=y_1],\cdots\cdots,[x_n=y_n])<A^+(w)$.

Then

$\lambda=\min(A(x_1),\cdots\cdots,A(x_n),A(y_1),\cdots\cdots,A(y_m))>0$

and $\min([n=m],[x_1=y_1],\cdots\cdots,[x_n=y_m]=0$, i.e.

$x_1,\cdots\cdots,x_n,y_1,\cdots\cdots,y_m\in A_\lambda$, and $m\ne n$ or $x_i\ne y_i$ for $1\le i\le n$ so, A_λ is not a code, this is a contradiction, thus A is a F-code.

Theorem2.4. If a F-set A of $F(X^*)$ is a F-code, then any morphism $\beta:Y^*\to X^*$ which induces a bijecton of some aphabet Y onto suppA is injective. Conversely, if there exists an injiective morphism $\beta:Y^*\to X^*$ such that $\sup pA=\beta(Y)$,then A is a F-code.

Proof. Let $\beta:Y^*\to X^*$ be a morphism such that β is a bijection of Y onto suppA. Let $u,v\in Y^*$ be words such that $\beta(u)=\beta(v)$. If $u=e$, then $v=e$; indeed $\beta(y)\ne e$ for each letter $y\in Y$, since A is a F-code. If $u\ne e$ and $v\ne e$,set $u=y_1y_2\cdots\cdots y_n$, $v=y'_1y'_2\cdots\cdots y'_m$ with $n,m\ge 1$, $y_1,\cdots\cdots,y_n,y'_1\cdots\cdots y'_m\in Y$. Since β is a morphism, we have $\beta(y_1)\cdots\cdots\beta(y_n)=\beta(y'_1)\cdots\cdots\beta(y'_m)$. But A is a F-code and $\beta(y_i), \beta(y'_j) \in \sup pA$, so

$\min(A(\beta(y_1)),\cdots\cdots,A(\beta(y_n)),A(\beta(y'_1)),\cdots,A(\beta(y_m)))$

$\propto \min([m=n],[\beta(y_1)=\beta(y'_1)],\cdots\cdots,[\beta(y_n)=\beta(y'_n)])$

$\geq A^+(\beta(u))>0$,

$\min([m=n],[\beta(y_1)=\beta(y'_1)],\cdots\cdots,[\beta(y_n)=\beta(y'_n)])>0$. Thus n=m and $\beta(y_i)=\beta(y'_i)$ for $i=1,\cdots\cdots,n$, Now β is injective on Y. Thus $y_i=y'_i$ for $i=1,\cdots\cdots,n$, and u=v. This shows that β is injective.

Conversely, if $\beta:Y^*\to X^*$ is an injective morphism ,for any $n,m\geq 1$, $x_1,\cdots\cdots,x_n$, $x'_1,\cdots\cdots,x'_m \in X^*$ and $x_1\cdots\cdots x_n = x'_1\cdots\cdots x'_m = x$ and $x_1,\cdots\cdots,x_n$, $x'_1,\cdots\cdots,x'_m \in \sup pA=\beta(Y)$, then we consider the letters y_i, y'_i in Y such that

$\beta(y_i)=x_i, \beta(y'_j)=x'_j$, $i=1,\cdots\cdots,n$,

$j=1,\cdots\cdots,m$, Since β is injective morphism,

$x_1\cdots x_n=x'_1\cdots x'_m \Rightarrow \beta(y_1)\cdots\beta(y_n)=\beta(y'_1)\cdots\beta(y'_m)$

$\Rightarrow \beta(y_1\cdots y_n)=\beta(y'_1\cdots y'_m)\Rightarrow$

$y_1\cdots\cdots y_n = y'_1\cdots\cdots y'_m$.Thus n=m and $y_i=y'_i$, since Y is code over ifself so $x_i=x_i$ for $i=1,\cdots\cdots,n$.

$\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m)) \propto$
$\min([m=n],[x_1=x'_1],\cdots\cdots,[x_n=x'_n]) =$
$\min(A(x_1),\cdots\cdots,A(x_n)) \propto 1 \geq A^+(x)$. Meanwhile if there exists x_i or $x'_j \notin \sup pA$, then

$\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m)) \propto$
$\min([m=n],[x_1=x'_1],\cdots\cdots,[x_n=x'_n]) = 0 \propto$
$\min([n=m],[x_1=x'_1],\cdots\cdots,[x_n=x'_n]) = 1 \geq$

$A^+(x)$. Therefore A is a F-code. Over X.

Definitoion2.1. A morphism $\beta:Y^*\to X^*$ which is imjective and such that $\sup pA=\beta(Y)$ is called a F-coding morphism for A. Where A is a F-code.

Corollary2.5. Let $\alpha:X^*\to Y^*$ be an injectiive morphism. If A is a F-code over X, then $\alpha(A)\in F(Y^*)$ which defined by $\sup p\alpha(A)=\alpha(\sup pA)$ is a F-code over Y. If B is a F-code over Y, then $\alpha^{-1}(B)$ which defined by $\alpha^{-1}(B)\in F(X^*)$ and $\sup p\alpha^{-1}(B)=\alpha^{-1}(\sup pB)$, is a F-code over X.

Proof. Let $\beta:Z^*\to X^*$ be a F-coding morphism for A. Then $\alpha(\beta(Z))=\alpha(\sup pA)$ and since $\alpha\cdot\beta:Z^*\to Y^*$ is an injective morphism. By theorem2.4 $\alpha(A)$ is a F-code.

Conversely, let $A=\alpha^{-1}(B)$, $n,m\geq 1$,

$x_1,\cdots\cdots,x_n$, $x'_1,\cdots\cdots,x'_m \in X^*$, $x_1x_2\cdots\cdots x_n =$ $x'_1\cdots\cdots x'_m = x$, then $\alpha(x_1)\alpha(x_2)\cdots\cdots\alpha(x_n)=$ $\alpha(x'_1)\cdots\cdots\alpha(x'_m)$, as B is a F-code, therefore $\min(\beta(\alpha(x_1)),\cdots\cdots,\beta(\alpha(x_n)),\beta(\alpha(x'_1)),\cdots\cdots,$ $\beta(\alpha(x'_m))) \propto$
$\min([m=n],[\alpha(x_1)=\alpha(x'_1)],\cdots\cdots,[\alpha(x_n)$

$=\alpha(x'_n)]) \geq B^+(\alpha(x))$. If $A^+(x)>0$, then

$x\in \sup pA^+ = [\sup p\alpha^{-1}(B)]^+ = \alpha^{-1}(\sup pB^+)$ and

it implies $\alpha(x)\in suup(B)^+$. So

$\min([m=n],[\alpha(x_1)=\alpha(x'_1)],\cdots\cdots,[\alpha(x_n)$ $=\alpha(x'_n)])=1$, i.e. m=n, $\alpha(x_i)=\alpha(x'_i)$, for $i=1,\cdots\cdots,n$. The injectivity of α implies that $x_i=x'_i$ for $i=1,\cdots\cdots,n$, so $\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m)) \propto$
$\min([m=n],[x_1=x'_1],\cdots\cdots,[x_n=x'_n]) =$
$\min(A(x_1),\cdots\cdots,A(x_n),A(x'_1),\cdots\cdots,A(x'_m)) \propto$

$1=1\geq A^{+}(x)$. If $A^{+}(x)=0$, it is obvious. Therefore A is a F-code.

Corollary 2.6. If $A\in F(X^{*})$ is a F-code, then A^{n} is a F-code for any interger $n>0$.

Proof. Let $\beta:Y^{*}\to X^{*}$ be a F-coding morphism for A, then $\sup pA^{n}=\beta(Y^{n})$. But Y^{n} is a F-code, Thus the conclusion follows from theoren2.4.

Definition2.2. A F-set $A\in F(X^{*})$ is called F-prefix(suffix) if for any $x,x',u\in X^{*}$, $x=x'u(x=ux')$ implies $\min(A(x),A(x'))\leq[x=x']$. And a F-set is F-biprefix if it is both F-prefix and F-suffix.

Theorom2.7. A is F-prefix (suffix, biprefix) set $\Leftrightarrow A_{\lambda}$ is ordinary prefix (suffix,biprefix) set for any $\lambda\in(0,1]$.

Proof. "$\Rightarrow$" For any $x,x',u\in X^{*},x=x'u$ since A is F-prefix set, we have $\min(A(x),A(x'))\leq[x=x']$. If $x,x',\in A_{\lambda}$, $\lambda\in(0,1]$, then $A(x)\geq\lambda, A(x')\geq\lambda$ and $[x=x']\geq\lambda>0$, thus $x=x'$ and A_{λ} is prefix set.

"$\Leftarrow$" In order to prove that A is F-prefix set, suppose the contrary, then there exists $x,x',u\in X^{*}$ such that $x=x'u$ and $\min(A(x),A(x'))>[x=x']$. So $0<\min(A(x),A(x'))\leq 1$ and $[x=x']<1$. Set $\lambda=\frac{\min(A(x),A(x'))}{2}\in(0,\frac{1}{2})$, it implies $x,x',\in A_{\lambda}$ and $x\neq x'$, i.e. A_{λ} is not a prefix set, which yields the contradiction. Hence A is a F-prefix set.

Proposition2.8. Any supset of a F-prefix set is a F-prefix set.

Proposition2.9. Any F-prefix (suffix, biprefix) set $A\neq\{(e,1)\}$ is a F-code.

Proof. If A is not a F-code, then there is a word w of minimal length having two factorizations $w=x_{1}x_{2}\cdots\cdots x_{n}=x_{1}'x_{2}'\cdots\cdots x_{m}'$, $x_{i},x_{j}\in X^{*}$ such that

$$A^{+}(w)>\min(A(x_{1}),\cdots\cdots,A(x_{n}),A(x_{1}'),\cdots\cdots,A(x_{m}'))$$

$\propto\min([m=n],[x_{1}=x_{1}'],\cdots\cdots,[x_{n}=x_{n}']$ it implies $A(x_{i})>0, A(x_{j}')>0$ $i=1,\cdots\cdots,n$, $j=1,\cdots\cdots,m$ and $\min([m=n],[x_{1}=x_{1}'],\cdots\cdots,[x_{n}=x_{n}'])=0$. Both x_{1},x_{1}' are nonempty, and since w has minimal length, then $[x_{1}=x_{1}']=0$, i.e. x_{1},x_{1}' are distinct. So $x=x'u$ or $x_{1}'=x_{1}v$ for $u,v\in X^{+}$. Thus $\min(A(x_{1}),A(x_{1}'))>0=[x_{1}=x_{1}']$, which contradicts that A is F-prefix. Therefore A is a F-code. The similar argument holds for F-suffixsets.

Theorem2.10. $A\in F(X^{*})$ is a F-prefix code (suffix code, biprefix code)$\Leftrightarrow$ A is a F-prefix set (suffix set, biprefix set) and $A\neq\{(e,1)\}$.

Proof. "$\Leftarrow$" For any $x\in X^{+}$, if $x=x'u$ for all $x',u\in X^{+}$, then $\min(A(x),A(x'))\leq[x=x']=0$, so $(A\cap AX^{+})(x)=\min(A(x)\sup\{A(x')\mid x'u=x$ *for all* $x',u\in X^{+}\})=0$, i.e. $A\cap AX^{+}=\phi$.

Thus A is a F-prefix code.

"⇒" For any $x,x',u\in X^*, x=x'u$, if $x=e$, then $x'=e$, $\min(A(x),A(x'))\le[x=x']=1$.

Now suppose that $(A\cap AX^+)(x)=$

$\min(A(x),\sup\{A(x')\mid x'u=x,\ x',u\in X^+\})=0$,

if $A(x)=0$, obviously $\min(A(x),A(x'))=0\le[x=x']$.

If $\sup\{A(x')\mid x'u=x,x',u\in X^+\}=0$. then $A(x')=0$ and $\min(A(x),A(x'))=0\le[x=x']=0$

Therefore A is a F-prefix set and $A\ne\{(e,1)\}$.

Theorem2.11. A is a F-prefix code (suffix code, biprefix code) $\Leftrightarrow A_\lambda$ is prefix (suffix, biprefix) code for any $\lambda\in(0,1]$.

Proof. It can be proved easily by theorem2.7 and theorm2.10.

Definition2.3 A F-code A is called a maximal F-code over X if A is not properly contained in any other F-code over X, that is, if $A\subset A'$, A' is F-code, then $A=A'$.

Proposition2.12. Any F-code A over X is contained in some maximal F-code over X.

Proof. Let $\overline{F}$ be the set of F-code over X containing A, ordered by set inclusion. To show that $\overline{F}$ contains a maximal element, it suffices to demonstrate, in view of Zorn's lemma, that any chain $\overline{G}$ (i.e., any totally ordered F-subset) in of admits a least upper bound in $\overline{F}$.

Consider a chain C of F-codes containing A, then $\hat{B}=\bigcup_{B\cup\overline{G}} B$ is the least upper bound of C. It remains to show that $\hat{B}$ is a F-code. For this, let $n,m\ge1$, $y_1,\cdots\cdots,y_n,y'_1,\cdots\cdots,y'_m\in X^*$ be such that $y_1y_2\cdots\cdots y_n=y'_1y'_2\cdots\cdots y'_m=w$. If $y_1,\cdots\cdots,y_n,y'_1,\cdots\cdots,y'_m\in\operatorname{supp}\hat{B}=\bigcup_{B\in\overline{G}}\operatorname{supp}B$,

then each of the y_i,y'_j belongs to a support set of F-code of the chain $\overline{G}$ and this detrmines n+m elements of $\overline{G}$. One of them, say D, contains all the others. Thus $y_1,\cdots\cdots,y_n,y'_1,\cdots\cdots,y'_m\in\operatorname{supp}D$, and since D is a F-code, then

$\min(D(y_1),\cdots\cdots,D(y_n),D(y'_1),\cdots\cdots,D(y'_m))$
$\propto\min([m=n],[y_1=y'_1],\cdots\cdots,[y_n=y'_n]$

$\ge D^+(w)>0$. We have n=m and $y_i=y'_i$ for $i=1,\cdots\cdots,n$. Since $D\subseteq\hat{B}$,

$\min(\hat{B}(y_1),\cdots\cdots,\hat{B}(y_n),\hat{B}(y'_1),\cdots\cdots,\hat{B}(y'_m))$

$\propto\min([m=n],[y_1=y'_1],\cdots\cdots,[y_n=y'_n]=1$

$\ge\hat{B}(w)$. ……⊗ Otherwise for ⊗ is obvious.

Therefore $\hat{B}$ is a F-code.

3.F-codes and F-submonoids

Proposition3.1. Let X be an alphabet. M is a F-submonoid of X^* and

$A=M-\{(e,1)\}-(M-\{(e,1)\})^2$. Then $A^*\subseteq M$

and suppA is a unigue minimal set of generators of suppM .(A is called almost minimal set of generators of M).

Proof. Set $Q=M-\{(e,1)\}$. First, we verify that suppA generates suppM, i.e. that $(\sup pA)^*=\sup pM$. Since $A\subset M$, then for any $x\in X^*$,

$$A^2(x)=\sup_{\substack{y,z\in X^*\\ yz=x}}\min(A(y),A(z))$$

$$\leq \sup_{\substack{y,z\in X^* \\ yz=x}} \min(M(y), M(z))$$

$\leq \sup_{\substack{y,z\in X^* \\ yz=x}} M(x) = M(x)$, i.e.. $A^2 \subset M$. Then we can show that $A^n \subset M$ by induction on $n \geq 0$. Thus $A^* \subseteq M$ and $\sup pA^* \subseteq \sup pM$. We prove that $\sup pA^* \geq \sup pM$ by induction on the length of words. Of course, $e \in \sup pA^*$. Let $m \in \sup pQ$. If $m \in \sup pQ^2$, then $m \in \sup pA$. Otherwise $m = m_1 m_2$ with $m_1, m_2 \in \sup pQ$ both strictly shorter than m.Therefore m_1, m_2 belong to $\sup pA^*$. Since $\sup pA^* = \bigcup_{n=0}^{\infty} \sup pA^n$ then there exist nature numbers l, k such that $A^l(m_1) > 0, A^k(m_1) > 0$, so

$$A^{l+k}(m) = \sup_{\substack{y,z\in X^* \\ yz=m}} \min(A^l(y), A^k(m))$$

$\geq \min(A^l(m_1), A^k(m_2)) > 0$, and

$A^*(m) \geq A^{l+k}(m) > 0$, i.e. $m \in \sup pA^*$.

Hence $(\sup pA)^* = \bigcup_{n=0}^{\infty} (\sup pA)^n = \bigcup_{n=0}^{\infty} \sup pA^n$ $= \sup pA^* = \sup pM$.

Now let B be F-set such that suppB is a set of generators of suppM. We may suppose that $(e,1) \notin B$. Then each $x \in \sup pA$ is in $\sup pB^*$ and therefore can be written as $x = y_1 y_2 \cdots\cdots y_n$ ($y_i \in \sup pB, n \geq 0$). The facts that $x \neq e$ and $x \neq \sup pQ^2$ force n=1 and $x \in \sup pB$. This shows that $\sup pA \subseteq \sup pB$. Thus $\sup pA$ is a minimal set of generators and such a set is unique.

Proposition3.3. If M is a F-free submonoid of X^*,then a F-set which support set is a minimal set of $\sup pM$ is a code. Conversely, if $A \in F(X^*)$ is a F-code, then the F-submonoid A^* of X^* is F-free and A is its minimal F-set of generators.

Proof. let $\alpha : Y^* \to \sup pM$ be an isomorphism. Then α considered. as a morphism from Y^* into X^*, is injective. By theorem2.4, the F-set A which such that $\sup pA = \alpha(Y)$ is a F-code. Next

$\sup pM = \alpha(Y^*) = (\alpha(Y))^* = (\sup pA)^*$. Thus $\sup pA$ generates $\sup pM$. Furthermore $Y = Y^+ - Y^+Y^+$ and $\alpha(Y^+) = \sup p(M - \{(e,1)\})$. Consequently $\sup pA = \sup p\{(M - \{(e,1)\} - (M - \{(e,1)\})^2\}$, showing that $\sup pA$ is the minimal set of generators of $\sup pM$.

Conversely, assume that $A \in F(X^*)$,is a F-code and consider a F-coding morphism $\alpha : Y^* \to X^*$ for A. Then α is injective and α is a bijection from Y into $\sup pA$. Thus α is a bijection from Y^* onto $\alpha(Y^*) = (\alpha(Y))^* = (\sup pA)^*$. Since A is a F-code, then A^* is a F-free. Now α is a bijection, thus $Y = Y^+ - Y^+Y^+$ implies $A = A^+ - A^+A^+$, showing by proposition3.1 that $\sup pA$ is the minimal set of generators of $\sup pA^*$. Since A generates A^*, thus A is the minimal F-set of generators of A^*.

Corollary3.3. Let A and B be F-codes over X. If $A^* = B^*$, then $A = B$.

Definition 3.1. A F-submonoid N of monoid S is F-stable if for all $u, v, w \in S$,
$\min(N(u), N(u), N(uw), N(wu)) > 0$ implies $N(w) > 0$.

Proposition3.4. A F-submonoid N of X^* is F-stable, then

$A = N - \{(e,1)\} - (N - \{(e,1)\})^2$ is a F-code.

Proof. To prove that A is a F-code, suppose the contrary. Then there is a word $z \in \sup pA^+$ of minimal length, having two distinct factorization in words of $\sup pA$. $z = x_1 x_2 \cdots\cdots x_n = y_1 y_2 \cdots\cdots y_m$ with $x_1, \cdots\cdots, x_n, y_1, \cdots\cdots, y_m \in \sup pA$. We may suppose $|x_1| < |y_1|$. Then $y_1 = x_1 w$ for some nonempty word w. Since N is F-stable, then $\min(N(x_1), N(y_2 \cdots\cdots y_m), N(XU), N(wy_2 \cdots\cdots y_m))$
$= \min(N(x_1), N(y_2 \cdots\cdots y_m), N(y_1), N(x_2 \cdots\cdots x_m)$
$\geq \min(A(x_1), \cdots\cdots, A(x_N), A(y_1), \cdots\cdots, A(y_M)) > 0$ (Since N is F-submonoid, $N(x_2 \cdots\cdots x_n) \geq$
$\min(N(x_2), \cdots\cdots, N(x_n)) = \min(A(x_2), \cdots\cdots, A(x_n))$.
So

$N(w) > 0$ and $(N - \{(e,1)\})^2 (x_1 w) = \sup_{\substack{u,v \in X^* \\ uv = x_1 w}}$

$\min(N(u), N(v)) \geq \min(N(x_1), N(w)) > 0$. Consequently $y_1 = x_1 w \notin \sup pA$, which yields the contradiction. Thus A is a F-code.

Propostion3.5. A F-free submonoid N of X^* is F-stable.

Proof. Since N is F-free, then let B be its F-base. Let $u, v, w \in X^*$ and suppose that $u, v, uw, wv \in \sup pN$. Set $u = x_1 \cdots\cdots x_k$, $wv = x_{k+1} \cdots\cdots x_r$, $uw = y_1 \cdots\cdots y_l$, $v = y_{l+1} \cdots\cdots y_s$ with x_i, y_j in $\sup pB$. The equality $u(wv) = (uw)v$ implies $x_1 \cdots\cdots x_k x_{k+1} \cdots\cdots x_r = y_1 \cdots\cdots y_l y_{l+1} \cdots\cdots y_s$. Since B is F-bused of N and N is a F-subnonoid,
$\min(B(x_1), \cdots\cdots, B(x_r), B(y_1), \cdots\cdots, B(y_s)) \propto$
$\min([r = s], [x_1 = y_1], \cdots\cdots, [x_r = y_s]) \geq N(uwv)$
$\geq \min(N(u), N(wv)) > 0$. It implies $r = s$ and $x_i = y_i$ $(i = 1, \cdots\cdots, s)$. Moreover, $l \geq k$ because $|uw| \geq |u|$. Showing that $uw = x_1 \cdots\cdots x_k x_{k+1} \cdots\cdots x_l = ux_{k+1} \cdots\cdots x_l$ hence $w = x_{k+1} \cdots\cdots x_l$. If $w \neq e$, $N(w) = B^*(w) \geq B^n(w)$
$\geq \min(B(x_{k+1}), \cdots\cdots, B(x_l)) > 0$. If $w = e$, $N(w) = 1$, thus N is stable.

Definition3.2. Let N be a F-submonoid of X^*. N is F-right (F-left) unitary if for all $u, v \in X^*$,

$\min(N(u), N(uv)) > 0$ ($\min(N(u), N(uv)) > 0$) implies $N(u) > 0$.

The F-submonoid N of X^* is F-biunitary if it is both F-left and F-right unitary.

Proposition3.6. A F-submonoid M of X^* is F-right (resp., left, bi-)unitary iff its almost minimal set of generators is a F-prefix(rewp, suffix, bijsefix) code.

Proof. Let $M - \{(e,1)\} = Q$ and $Q - Q^2 = A$ be its almost minimal set of generators. Suppose M is F-right unitary.

To show that A is F-prefix, let x, xu be in $\sup pA$ for some $u \in X^*$. Then $\min(M(x), M(xu)) = \min(A(x), A(xu)) > 0$. Since M is F-right unitary, $M(u) > 0$. If $u \neq e$, then $u \in \mathrm{supp}Q$, but then $xu \in \mathrm{supp}Q^2$

contrary to the assumption. Thus $u=e$ and $\min(A(x),A(xu))\le[x=xu]$. i.e. A is F-prefix.

Conversely, suppose A is F-prefix. Let $u,v\in X^*$ be such that $u,uv\in \sup pM=(\sup pA)^*$. Then $u=x_1x_2\cdots\cdots x_n, uv=y_1y_2\cdots\cdots y_m$ for some $x_1,\cdots\cdots,x_n,y_1,\cdots\cdots,y_m\in \sup pA$. Consequently $x_1x_2\cdots\cdots x_nv=y_1y_2\cdots\cdots y_m$. Since A is F-prefix, neither x_1 nor y_1 is a proper left facter of the other. Thus $x_1=y_1$, and for the some reason $x_2=y_2,\cdots\cdots,x_n=y_n$. This shows that $m\ge n$ and $v=y_{n+1}\cdots\cdots y_m$ belongs to $\sup pM$. Thus M is F-right unitary.

Proposition3.7. If M is a maximal F-free submoniod of X^*, then its F-base A is a maximal F-code.

Proof. It is obvious that A is a F-code. Let B be a F-code on X with $A\subset B$ and $A\ne B$. Then $A^*\subset B^*$ and $A^*\ne B^*$ since otherwise $A=B$ by corollary 3.3. Now A^* is maximal. Thus $B^*=X^*$ and $B=X$. Thus $A\subset X^*$ and $A\ne X$. Let $b\in X-A$. The F-set $C=A\cup b^2$ is a F-code and $M=A^*\not\subseteq C^*\not\subseteq X^*$ since $b^2\in\sup pM$ and $b\notin\sup pZ^*$. This contradicts the maximality of M.

Theorem3.8. The almost minimal set of generators of the intersection of an arbitrary family of F-free submonids of X^* is a F-code.

Proof. Let $(Mi)_{i\in I}$ be a family of F-free submonoids of X^*, and set $M=\bigcap_{i\in I}M_i$. Clearly M is a F-submonoid, and it suffices to show that M is F-stable. If for all $u,v,w\in X^*$, and $u,v,uw,wv\in\sup pM$, then these four words belong to each of the $\sup pM_i$. Since each M_i is F-free, by proposition3.5, each M_i is F-stable, w is in $\sup pM_i$ for each $i\in I$. Thus $w\in\sup pM$. By proposition3.6, the almost minimal set of generators of M is a F-code.

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