Selfconjugate fuzzy implications*

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Abstract

Paper deals with singular classes of similar fuzzy implications and describes the lattice of all invariant fuzzy implications.

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1 Introduction

In fuzzy set theory we deal with binary operations in unit interval as triangular norms, triangular conorms and fuzzy implications. Equivalence classes of such operations are characterized by similarity relation. Similar elements are constructed by formula

$$T^*(x,y) = T^*_\varphi(x,y) = \varphi^{-1}(T(\varphi(x),\varphi(y))), \qquad \text{for } x,y \in [0,1],$$

where φ is an increasing bijection on unit interval. For special operations such constructed T^* element coincides with T. For example t-norm minimum or s-norm maximum give

$$min^* = min, max^* = max.$$

The problem arises about existence and properties of such selfsimilar operations. Here we consider this problem for fuzzy implication.

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2 Fuzzy implications

Fuzzy implications are described and examined in many ways (cf. Baldwin, Pilsworth [4], Cordón, Herrera, Peregrin [5], Dubois, Prade [6], Fodor, Roubens [8] or Kiszka, Kochańska, Śliwińska [11]). We use here the simplest definition of fuzzy implication presented by Fodor and Roubens [8].

Definition 1. Any function $I\colon [0,1]^2 \to [0,1]$ is called *fuzzy implication* if it fulfils the following conditions:

I1. $\forall_{x,y,z\in[0,1]}\ (x \leqslant z \Rightarrow I(x,y) \geqslant I(z,y))$,

I2. $\forall_{x,y,z\in[0,1]}\ (y \leqslant z \Rightarrow I(x,y) \leqslant I(x,z))$,

I3. $\forall_{y\in[0,1]}\ I(0,y) = 1$,

I4. $\forall_{x\in[0,1]}\ I(x,1) = 1$,

I5. $I(1,0) = 0$.

Set of all fuzzy implications will be denoted by FI.

The cited notion of fuzzy implication fulfils two important condition: Firstly, it is a generalization of classical implication, it means that I fulfils the implication truth table:

$$I(0,0) = I(0,1) = I(1,1) = 1, I(1,0) = 0. \tag{1}$$

Secondly, I is decreasing with respect to the first variable and increasing with respect to the second variable. Using additional assumptions we can characterize similarity classes of fuzzy implications (cf. Smets, Magrez [13] or Baczyński [1]).

Definition 2 ([10], Chapter 8). Fuzzy implications $I, J \in FI$ are conjugate if there exists a bijection $\varphi\colon [0,1] \to [0,1]$ such that $J = I^*_\varphi$, where

$$I^*(x,y) = I^*_\varphi(x,y) = \varphi^{-1}(I(\varphi(x),\varphi(y))), \qquad \text{for } x,y \in [0,1]. \tag{2}$$

Let Φ denote the family of all increasing bijections φ: $[0,1] \rightarrow [0,1]$ and $I, J \in FI$. Fuzzy implication J is Φ-conjugate with I if

$$\exists_{\varphi \in \Phi} \left(J = I^*_\varphi\right). \tag{3}$$

Fuzzy implication I is called selfconjugate (Φ-selfconjugate) if

$$\forall_{\varphi \in \Phi} \left(I = I^*_\varphi\right). \tag{4}$$

Our problem is to desribe family of all selfconjugate fuzzy implications.

3 Invariant domains

We need an auxiliary notion of invariant set or invariant domain. It is invariant with respect to the family of bijections. However this notion is too general. There exist over two thousands different invariant domains. So we consider a special case: minimal invariant domains. It is such nonempty domain, which doesn't have a proper subdomain.

Definition 3. Set $D \subset [0,1]^2$ is invariant with respect to Φ (invariant domain) if $F(D) = D$, where
$F(x,y) = (\varphi(x), \varphi(y)), \varphi \in \Phi, x, y \in [0,1]$.
An invariant domain is minimal if it does not contain a proper subdomain.

The first important result describes minimal invariant domains.

Theorem 1. *There exist exactly 11 minimal invariant domains in* $[0,1]^2$*:*
$D_1 = \{(0,1)\}, D_2 = \{(0,1)\}, D_3 = \{(1,0)\}, D_4 = \{(1,1)\}$,
$D_5 = \{(x,x) : x \in (0,1)\}, D_6 = \{0\} \times (0,1), D_7 = \{1\} \times (0,1)$,
$D_8 = (0,1) \times \{0\}, D_9 = (0,1) \times \{1\}$,
$D_{10} = \{(x,y) : x \in (0,1),\ 0 < y < x\}$,
$D_{11} = \{(x,y) : x \in (0,1),\ x < y < 1\}$.
Every invariant domain is a sum of minimal domains.

Minimal invariant domains are numbered in such order: first, we have four single domains, next, we have five segment domains, last, we have two triangle domains. They are depicted in symbolical Figure 1.

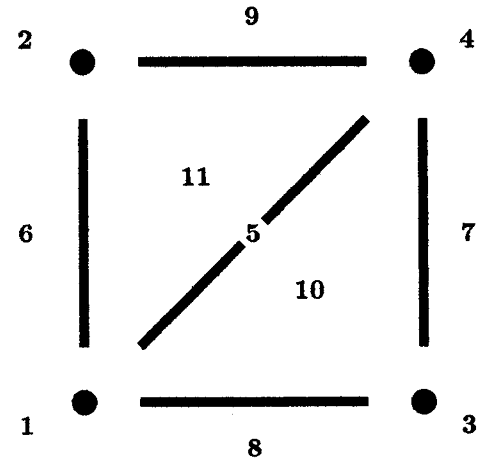

Fig.1. Minimal invariant domains $D_1 - D_{11}$.

4 Invariant implications

Now we return to fuzzy implications.

Theorem 2. *Family of all selfconjugate fuzzy implications is finite and consist of 18 implications* $I_1 - I_{18}$ *of the form:*

$$I_k(x,y) = \begin{cases} 1 & , \text{ if } x \in A_k \\ y & , \text{ if } x \in B_k \ , \\ 0 & , \text{ if } x \in C_k \end{cases} \tag{5}$$

where $x, y \in [0,1]$, *and* $A_k, B_k, C_k, k = 1, 2, ..., 18$ *are the invariant domains.*

Theorem 2 brings two important results. Firstly, we see that the family of such implications is finite. Secondly, selfconjugate implication on invariant domain can be constant or equal to identinty with respect to the second variable. It is the meaning of formula (5). Exact description of invariant domains A, B and C can be seen from full list of constructed implications:

$$I_1(x,y) = \begin{cases} 1 & , \text{ if } x < 1 \text{ or } y > 0 \\ 0 & , \text{ if } x = 1 \text{ and } y = 0 \end{cases}, \tag{6}$$

$$I_2(x,y) = \begin{cases} 1 & , \text{ if } x < 1 \\ y & , \text{ if } x = 1 \end{cases}, \tag{7}$$

$$I_3(x,y) = \begin{cases} 1 & , \text{ if } x = 0 \text{ or } y > 0 \\ 0 & , \text{ if } x > 0 \text{ and } y = 0 \end{cases}, \tag{8}$$

$$I_4(x,y) = \begin{cases} 1 & , \text{ if } x < 1 \text{ or } y = 1 \\ 0 & , \text{ if } x = 1 \text{ and } y < 1 \end{cases}, \tag{9}$$

$$I_5(x,y) = \begin{cases} 1 & , \text{ if } x < 1 \text{ and } y > 0 \text{ or } x = 0 \\ y & , \text{ if } x = 1 \\ 0 & , \text{ if } x > 0 \text{ and } y = 0 \end{cases}, \tag{10}$$

$$I_6(x,y) = \begin{cases} 1 & , \text{ if } x < 1 \text{ and } y > 0 \text{ or } x = 0 \text{ or } y = 1 \\ 0 & , \text{ if } x = 1 \text{ and } y < 1 \text{ or } x > 0 \text{ and } y = 0 \end{cases}, \tag{11}$$

$$I_7(x,y) = \begin{cases} 1 & , \text{ if } x \leqslant y \\ y & , \text{ if } x > y \end{cases}, \tag{12}$$

$$I_8(x,y) = \begin{cases} 1 & , \text{ if } x \leqslant y \\ y & , \text{ if } y < x < 1 \\ 0 & , \text{ if } x = 1 \text{ and } y < 1 \end{cases}, \tag{13}$$

$$I_9(x,y) = \begin{cases} 1 & , \text{ if } x < y \text{ or } x = 0 \\ y & , \text{ if } y \leqslant x \text{ and } x > 0 \end{cases}, \tag{14}$$

$$I_{10}(x,y)=\begin{cases}1 & , \text{ if } x\leqslant y\\ 0 & , \text{ if } x>y\end{cases}, \tag{15}$$

$$I_{11}(x,y)=\begin{cases}1 & , \text{ if } x<y \text{ or } x=0 \text{ or } y=1\\ y & , \text{ if } y\leqslant x \text{ and } 0<x<1\\ 0 & , \text{ if } x=1 \text{ and } y<1\end{cases}, \tag{16}$$

$$I_{12}(x,y)=\begin{cases}1 & , \text{ if } x=0\\ y & , \text{ if } x>0\end{cases}, \tag{17}$$

$$I_{13}(x,y)=\begin{cases}1 & , \text{ if } x<y \text{ or } x=0\\ y & , \text{ if } x=y \text{ and } x>0\\ 0 & , \text{ if } x>y\end{cases}, \tag{18}$$

$$I_{14}(x,y)=\begin{cases}1 & , \text{ if } x=0 \text{ or } y=1\\ y & , \text{ if } 0<x<1\\ 0 & , \text{ if } x=1 \text{ and } y<1\end{cases}, \tag{19}$$

$$I_{15}(x,y)=\begin{cases}1 & , \text{ if } x<y \text{ or } x=0 \text{ or } y=1\\ 0 & , \text{ if } x>y \text{ or } x=y \text{ and } 0<x<1\end{cases}, \tag{20}$$

$$I_{16}(x,y)=\begin{cases}1 & , \text{ if } x=0\\ y & , \text{ if } 0<x\leqslant y\\ 0 & , \text{ if } x>y\end{cases}, \tag{21}$$

$$I_{17}(x,y)=\begin{cases}1 & , \text{ if } x=0 \text{ or } y=1\\ y & , \text{ if } 0<x<y<1\\ 0 & , \text{ if } x>y \text{ or } x=y \text{ and } 0<x<1\end{cases}, \tag{22}$$

$$I_{18}(x,y)=\begin{cases}1 & , \text{ if } x=0 \text{ or } y=1\\ 0 & , \text{ if } x>0 \text{ and } y<1\end{cases}, \tag{23}$$

where $x,y\in[0,1]$.

5 Lattice of implications

Now we examine order relation between invariant implications.

Theorem 3 ([2]). *Set of all selfconjugate fuzzy implications is a distributive lattice.*

Theorem 4. *Selfconjugate fuzzy implications (6)-(23) form the lattice presented in Fig.2. This lattice is generated by implications:* I_3, I_4, I_{10}, I_{12}, I_{15} *and* I_{18}.

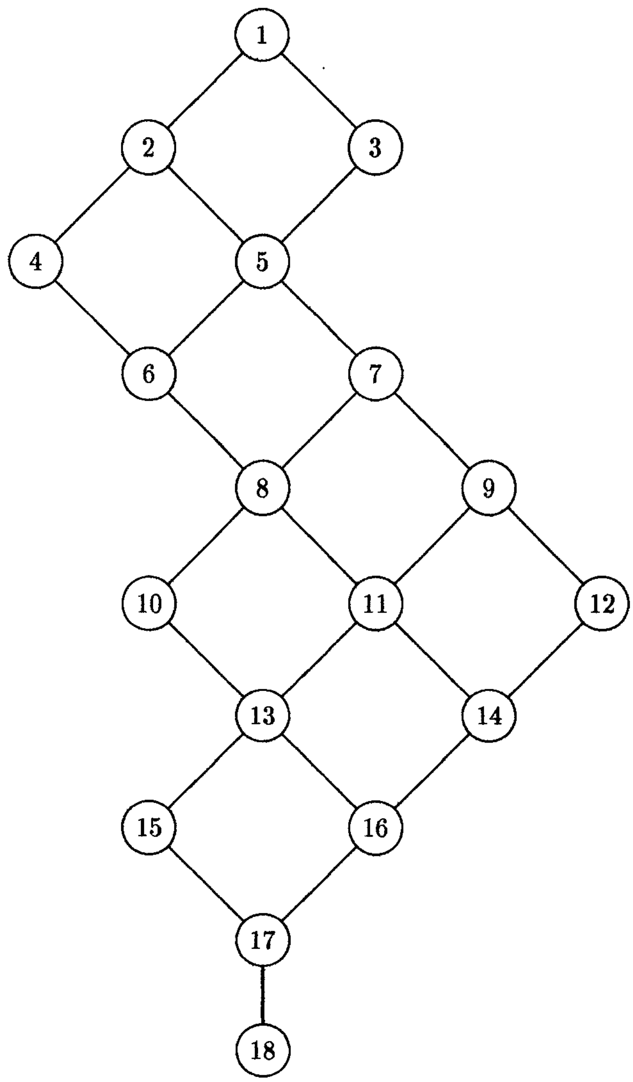

Fig.2. Lattice of fuzzy implications (6)-(23).

The above numbering of invariant implications is connected with Hasse diagram of this lattice. Implications are numbered from the top to the bottom, and from the left to the right. So I_1 is the greatest element, and I_{18} is the least element.

6 Contrapositive implications

An important class of fuzzy implications is distinguished by the contraposition principle.

Definition 4 ([8]). Implication $I \in FI$ is called contrapositive if

$$I(1-y, 1-x) = I(x,y), \qquad x,y \in [0,1]. \tag{24}$$

Theorem 5. *Contrapositive elements of the lattice from Theorem 3 are all comparable and form the chain:*

$$I_{18} \leqslant I_{15} \leqslant I_{10} \leqslant I_6 \leqslant I_1. \tag{25}$$

7 Remarks

Family of the above presented fuzzy implications will be examined later in connection with fuzzy logic and approximate reasoning. Using equivalence relation (3) we obtain conjugacy classes of fuzzy implications and condition (4) describes singular classes (cf. [3]).

Remark 1. Some of constructed fuzzy implications were presented in papers on fuzzy and multivalued logic. E.g. implications I_1 and I_2 appeared in [7], I_4 was presented in [5], I_7 is Gödel's implication [9] and I_{10} was presented by Rescher [12], p.47.

From the measure point of view, change of values on single element domain, or in segment domain is not important. From this point of view e have only 6 important models of invariant implications.

Definition 5. Fuzzy implications $I, J \in FI$ are near if they coincide on D_{10} and D_{11}. Then we write $I \sim J$.

Remark 2. There are six classes of invariant implications in the sense of Definition 5. Namely
$I_1 \sim I_2 \sim I_3 \sim I_4 \sim I_5 \sim I_6$,
$I_7 \sim I_8 \sim I_9 \sim I_{11}$, $I_{10} \sim I_{13} \sim I_{15}$,
$I_{12} \sim I_{14}$, $I_{16} \sim I_{17}$, I_{18}.
From this point of view only implicationsI_{12},I_{16} and I_{18} bring new models (cf. Remark 1).

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