Finite Power Property of Fuzzy Regular Language*

Bai Mingqiang MoZhiwen
(Department of Mathematics of Sichuan Normal University, Chengdu 610066)

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Abstract: In this paper the finite power property of fuzzy finite regular language has been introduced,and a serial of properties about it have been obtained.

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0 Introduction

Since E.T.Lee and L.A.Zadeh have given the concept of fuzzy finite-state automaton in 1969[1],there are some researches about it. But the researches about fuzzy regular language are little because it is difficult and challenging. In this paper the concept of finite power property of fuzzy regular language is given at first and the researches about it have been done preliminaryly.

1 Basic concepts

Definition 1.1 A fuzzy regular language L possesses the finite power property (in short,FP) if the set

$$\{L^i | i = 0, 1, 2, ...\}$$

is finite.

Definition 1.2 The order of a fuzzy regular language L is the smallest integer k satisfying $L^k = L^{k+1}$, if no such k exists we say the order of L is ∞.

The following simple lemma deals with the interconnections between the notions just defined.

Lemma 1.3 The following conditions (i)-(iii) are equivalent for a nonempty fuzzy regular language L:(i) L possesses FP;(ii)there is an integer k such that L is of order k;(iii)there is an integer k such that $L^k = L^*$.

Proof:First observe that each of the conditions (i)-(iii) implies that $(\lambda, 1) \in L$ because,otherwise,for all integer i, the shortest word in L^i is shorter than the shortest

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word in L^{i+1}.Hence,each of the conditions (i)-(iii) immediately follows.Observe,in particular,that condition (i) implies that $L^k = L^{k+i}$ hold for all integer i.

Every fuzzy star language,i.e. a fuzzy regular language of the form L^*,possesses FP.In fact the order of such a language equals 1.

Definition 1.4 Let G is a fuzzy finite automaton,the iterate G^* of G is the following finite automaton,where the set of states of G^* equals that of G added with a state denoted by I(for 'iterate'),the initial and final states in G^* equals the corresponding items in G.All transitions of G are presented also in G^* and G^* has the following transition to and from the new state I.

Whenever there is a transition in G from a state q to a final state, labeled by a and membership μ of a,then there is a transition in G^* labeled by a and μ from q to I.Whenever there is a transition labeled by a and μ in G from the initial state to a state q,then there is a transition in G^* labeled by a and μ from I to q.

Lemma 1.5 $(L(G))^* = L(G^*) \cup \{(\lambda, 1)\}$

Proof: The result is an immediate consequence of the definition of G^*.Consider a word $\omega \neq \lambda$ such that $(\omega, \mu) \in (L(G))^*$.It can be written as

$$\omega = \omega_1\omega_2 ... \omega_n$$

where $(\omega_i, \mu_i) \in L(G)$ for i=1,2,...,n and $\mu = \mu_1 \bigwedge \mu_2 \bigwedge \cdots \bigwedge \mu_n$.For $i < n$,there is a path in G^* from the initial state to I labeled by $\omega_1\omega_2...\omega_i$ and $\mu_1 \bigwedge \mu_2 \bigwedge \cdots \bigwedge \mu_i$,and consequently, we can start all over again in G,showing that $(\omega, \mu) \in L(G^*)$.

On the other hand,whenever there is a path in G^* from the initial state to I labeled by ω and μ,by the definition of G^*,then $(\omega, \mu) \in (L(G))^*$.By the definition of G^*,every word in $L(G^*)$ is in $(L(G))^*$.It is obvious that $(\lambda, 1) \notin L(G^*)$.Hence $(L(G))^* = L(G^*) \cup \{(\lambda, 1)\}$.

2 Main Results

Definition 2.1 The iteration number of a word ω such that $(\omega, \mu) \in L(G^*)$,in symbols $IN(\omega)$,is the smallest integer k such that there is a path labeled by ω and μ from the initial state to a final state in G^*,and passing k times through the state I.

Lemma 2.2 Let ω is nonempty word with membership μ in $(L(G))^*$ and $IN(\omega) = k$,then ω is in $(L(G))^{k+1}$ but not in any $(L(G))^i$ with $i < k+1$. Proof: By the definition of G^*,we can be expressed as $\omega = \omega_1\omega_2...\omega_{k+1}$ where $(\omega, \mu) \in (L(G))^*, (\omega_i, \mu_i) \in L(G)$ and $\mu = \bigwedge_{i=1}^{k+1} \mu_i$.Thus $(\omega, \mu) \in (L(G))^{k+1}$.

Assume ω can be written as $\omega = y_1y_2...y_i, i < k+1$, where $(\omega, \mu) \in (L(G))^*$, $(y_j, \mu_j) \in L(G)$ and $\mu = \bigwedge_{j=1}^{i} \mu_j$.Because G represents L(G),we obtain by the former a path in G^* from the initial to a final state and labeled by ω and μ but passing only i-1 times through the state I.Since $i-1 < k$,this condition contradicts the fact that $IN(\omega) = k$.

Remark:Because G^* is nondeterministic,given a word ω,there may be several paths starting with the initial state of G^* and labeled by ω and membership μ of ω.

Let START(ω) be the set of states $q \neq I$ in G^* such that there is a path in G^* from the initial state to q labeled by ω and μ,and let END(ω) be the set of all states $q \neq I$ in G^* such that there is a path in G^* from q to one of the final state labeled by ω and μ.

Lemma 2.3 Assume that a nonempty word ω in $(L(G))^*$ can be decomposed as $\omega = \omega_1\omega_2$,then START($\omega_1$)$\cap END(\omega_2)$ is nonempty.Conversely,whenever START(ω_1)$\cap$ $END(\omega_2)$ is nonempty for a word $\omega = \omega_1\omega_2$,then ω is in $(L(G))^*$.

Proof:The result is an immediate consequence of lemma 1.5 and the definition of START(ω) and END(ω).

Lemma 2.4 Assume that a nonempty word $\omega = xyz$ satisfies $(\omega,\mu) \in (L(G))^*$ such that START(x)=START(xy) and END(z)=END(yz),then if L(G) possesses FP,there are states q and q' in START(x)$\cap$ END(z) such that there is a path in G from q to q' labeled by y and membership μ_2 of y.

Proof:By lemma 2.3 and the given conditions,the intersection START(x) $\cap$ END(z) is nonempty.

Assuming that whenever q and q' are in START(x)$\cap$ END(z), there is no path from q to q' labeled by y and μ_2. It is a consequence of the known conditions that

$$START(x) = START(xy^i) \text{ and } END(z) = END(y^jz) \text{ for all } i \text{ and } j.$$

This fact implies by lemma2.3 that $(xy^iz,\mu) \in (L(G))^*$ for every i, where $(x,\mu_1) \in (L(G))^*, (z,\mu_3) \in L(G)$ and $\mu = \mu_1 \bigwedge \mu_2 \bigwedge \mu_3$.

On the other hand,our assumption implies that,for all i,IN(xy^iz) $\geq i$.But this fact and lemma 2.2 imply that for any k,we can find a word in the $(L(G))^*$ which is not in $(L(G))^k$.This conclusion contradicts the fact that L(G) possesses FP.

For the following proof, now we define some new things.We use #S to denote the cardinality of a finite set S.The order of a word ω in $(L(G))^*$ is defined by $o(\omega) = min\{k|(\omega,\mu) \in (L(G))^*\}$.For $(\omega,\mu) \in (L(G))^*$,we define $PAIRS(\omega) = \{(S,S')|\omega = xy, START(x) = S, END(\omega) = S'$,for some x and y(possible empty)$\}$ and WORST(ω)=(i,j),where $i = max\{$#S+#S'$|(S,S')$ in PAIRS(ω)$\}$,j=#{(S,S') in PAIRS(ω)|#S+#S' $= i\}$. Let b_n be the greatest number among the binomial cofficients C_{2n}^i and $o(i,j) = max\{o(\omega)|(\omega,\mu) \in (L(G))^*$ and $WORST(\omega) \leq (i,j)\}, o(i) = o(i,b_n)$.

Lemma 2.5 Assume that L(G) possesses FP and that i≥ 3 and j ≥ 3 are numbers satisfying $2 \leq i \leq 2n$ and $1 \leq j \leq b_n$.Then

$$o(i,j) \leq o(i-1) + o(i,j-1) + 3$$

Proof: If there is no word $(\omega,\mu) \in (L(G))^*$ such that WORST(ω) $= (i,j)$,then o(i,j)=o(i,j-1).Consequently the conclusion is satisfied.

Hence,to prove the lemma,we need to show that an arbitrary ω with property WORST(ω) $= (i,j)$ satisfies

$$o(i,j) \leq o(i-1) + o(i,j-1) + 3$$

Let $\omega = xz_1$ where $(\omega,\mu) \in (L(G))^*, (x,\mu_1) \in (L(G))^*, (z_1,\mu_2) \in L(G), \mu = \mu_1 \bigwedge \mu_2$,#(START(x))+#(END(z_1))=i and x is the shortest prefix of ω with this property.We denote $START(x) = S, END(z_1) = S'$.We next write $z_1 = yz$ where $(y,\mu_1') \in L(G), (z,\mu_2') \in L(G), \mu_2 = \mu_1' \bigwedge \mu_2', START(x) = START(xy) = S$ and

$END(yz) = END(z) = S'$ and z is the shortest suffix of z_1 with this property.Thus $\omega = xyz$. It is possible that one or two of words x,y and z equal λ.

By lemma2.4,there are states q and q' in $S \cap S'$ such that there is a path from q to q' labeled by y and μ_1' in G. Now we extend this path to a path from the initial state to a final one as follows.

Let u be a suffix of x such that there is a path in G from the initial state to q labeled by u and μ_3',where $(\omega_1, \mu_3'') \in L(G), (u, \mu_3') \in L(G)$ and $\mu_1 = \mu_3' \bigwedge \mu_3''$. Let v be a prefix of z such that there is a path in G from q' to a final state by v and μ_4' where $(v, \mu_4') \in L(G), (\omega_2, \mu_4'') \in L(G)$ and $\mu_2' = \mu_4' \bigwedge \mu_4''$.Thus $x = \omega_1 u, z = v\omega_2, \omega = \omega_1 uyv\omega_2$ for some words ω_1 and ω_2 in $(L(G))^*$.The existence of u and v is guaranteed by the fact that ω is in $(L(G))^*$.Moreover,$(uyv, \mu_3' \bigwedge \mu_1' \bigwedge \mu_4') \in L(G)$. It is possible that one or both of the words u and v are empty.

Because $(uyv, \mu_3' \bigwedge \mu_1' \bigwedge \mu_4') \in L(G)$ and $o(\omega' \omega'') \leq o(\omega') + o(\omega'')$, we obtain $o(\omega) \leq o(\omega_1) + 1 + o(\omega_2)$.

If $u \neq \lambda, o(\omega_1) \leq o(i-1)$; If u=$\lambda$ and $\omega_1 \neq \lambda$, we write x=$\omega_1 = \omega_3\omega_4$, where $(\omega_3, \mu_3) \in (L(G))^*$, $(\omega_4, \mu_4) \in L(G)$ and $\mu_1 = \mu_3 \bigwedge \mu_4$. This shows that $o(\omega_1) \leq o(i-1) + 1$,then if $\omega_1 = \lambda$,it is hold that $o(\omega_1) \leq o(i-1) + 1$.

If $v \neq \lambda, o(\omega_2) \leq o(i, j-1)$; If $v = \lambda$ and $\omega_2 \neq \lambda$,we write $\omega_2 = \omega_5\omega_6$,where $(\omega_5, \mu_5) \in (L(G))^*, (\omega_6, \mu_6) \in L(G)$ and $\mu_2' = \mu_5 \bigwedge \mu_6, \omega_5 \neq \lambda$,then $o(\omega_2) \leq o(i, j-1) + 1$.This hold when $\omega_2 = \lambda$.

Hence the lemma holds.

Lemma 2.6 Assume that L(G) possesses FP and that $i \geq 3$ satisfies $2 \leq i \leq n$.Then $o(i, 1) \leq o(i-1) + 3$.

Proof:When consiodering ω_2 in the lemma 2.5,o(i-1) rather than o(i,j-1) appears in the upper bound.Hence the lemma is previously hold.

Lemma 2.7 If L(G) posseses FP,then $o(2) \leq 1$.

Proof:If there are no words $(\omega, \mu) \in (L(G))^*$ such that the first component in the $WORST(\omega)$ equals 2,the result o(2)=1 by the definition.

Consider an arbitrary $(\omega, \mu) \in (L(G))^*, \omega \neq \lambda$ such that the first component in $WORST(\omega)$ equals 2.This condition implies by lemma 2.3 that,for any decomposition $\omega = \omega_1\omega_2$, where $(\omega_1, \mu_1) \in (L(G))^*, (\omega_2, \mu_2) \in L(G), \mu = \mu_1 \bigwedge \mu_2, START(\omega_1) \cap END(\omega_2)$ consists of exactly one state.Consequently,for any path p labeled by ω and μ in G^* starting from the initial state and passing through the state I,there is a path labeled by ω and μ in G starting from the initial state and ending at the same state as p. This means that $(\omega, \mu) \in L(G)$ and hence $o(\omega) = 1$.Since $o(\lambda) = 0$,the lemma holds.

Theorem 2.8 Assume that L(G) is a fuzzy regular language represented by a finite deterministic automaton with n states.Then there is an integer c_n effectively computable from n such that L(G) possesses FP if and only if $(L(G))^{C_n} = (L(G))^*$.Consequently,the FP problem is decidable.Moreover there is an algorithm for determining the order of a given fuzzy regular language.

Proof: Assume that L(G) possesses FP,by lemma 2.5,for any i such that $3 \leq i \leq 2n$.

$$\begin{aligned} o(i) = o(i, b_n) &\leq o(i-1) + 3 + o(i, b_n - 1) \\ &\leq 2(o(i-1) + 3) + o(i, b_n - 2) \leq \ldots \\ &\leq (b_n - 1)(o(i-1) + 3) + o(i, 1) \end{aligned}$$

$\leq (b_n + 1)(o(i-1) + 3) \leq 4(b_n + 1)o(i, 1)$

Now we denote $a_n = 4(b_n + 1)$ and $c_n = a_n^{2n-2}$.Consequently for any $3 \leq i \leq n, o(i) \leq a_n o(i-1)$.Hence

$$o(2n) \leq a_n o(2n-1) \leq a_n^2 o(2n-2) \leq ... \leq a_n^{2n-2} o(2) = c_n o(2) \leq c_n$$

Since the order of every word $(\omega, \mu) \in (L(G))^*$ at most o(2n),then $(L(G))^{C_n} = (L(G))^*$ .Consequently,we can solve the FP problem as follows. Given L(G),we first compute C_n.we then test whether or not $(L(G))^{C_n} = (L(G))^*$ hold. The order of a given fuzzy regular language L(G) can be determined as follows.We first decide by the method given above whether or not L(G) possess FP. If L(G) does not possess FP,it is of infinite order by lemma 1.3,otherwise,we test for$k = 0, 1, 2, ...$whether or not $(L(G))^k = (L(G))^{k+1}$. This sequence of tests terminates because L(G) possesses FP. The order of L(G) is the smallest k satisfying $(L(G))^k = (L(G))^{k+1}$.

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