N(2,0) Algebra and Lattice Implication Algebra

Jiang Zhaolin

(Department of Mathematics Lin yi Teachers College Shandong 276005, P. R. China)

Wu Duoyi

(Department of Mathematics Fo Shan Education College Guangdong 528000, P. R. China)

Abstract: This paper continues our investigation on lattice implication algebra. We discuss N(2,0) algebra and its proper subclass—Strong N (2, 0) algebra, and give a condition that strong N(2,0) algebra form lattice implication algebra.

Keywords: N(2,0) algebra, Strong N(2,0) algebra, Lattice implication algebra.

N(2,2,0) algebra is a algebraic system with two dual semi-group. In order to find the relation between it and LIA, we consider the consistence of $*$ and $\triangle$, then call N(2,2,0) algebra as N(2,0) algebra, in symbol, $(S, *, 0)$.

1. Preliminaries

Definition 1.1[1] Let S be a set with constant 0, two binary operations $*$ and $\triangle$ satisfy the following axioms:

(F_1) $x*(y\triangle z)=z*(x*y)$

(F_2) $(x \triangle y) * z = y * (x * z)$

(F_3) $0 * x = x$

then (S, * ,△,0) is called a N(2,2,0) algebra.

In N(2,2,0) algebra, it is easy to see that (S, * ,0) and (S,△,0) are two dual semigroup, in which there are two quasi—orders induced by * and △ respectively. Whenever * and △ are consistent, we have:

Definition 1.2 Let S be a set with costant 0, and the binary operation * satisfies:

(F_1') $x * (y * z) = z * (x * y)$

(F_2') $(x * y) * z = y * (x * z)$

(F_3') $0 * x = x$

for any $x, y, z \in S$, then (S, * ,0) is called a N(2,0) algebra.

Theorem 1.1 Let (S, * ,0) be a N(2,0) algebra, then the following ientities hold for any $x, y, z \in S$

(1) $x * y = y * x$

(2) $(x * y) * z = x * (y * z)$

(3) $x * (y * z) = y * (x * z)$, $(x * y) * z = (x * z) * y$

(4) 0 is unit element

So N(2,0) algebra is a commutative monoid.

From theorem 1.1 and definition 1.2, we have:

Theorem 1.2 Let (S, * ,0) be a N(2,0) algebra, then:

(F_1'), (F_2') and (F_3') iff (F_3'), (1) and (2).

Proof: ⇒Obvious.

⇐ Since $x * (y * z) = x * (z * y) = (x * z) * y = (z * x) * y = z * (x * y)$ so (F_1') hold. And since $(x * y) * z = x * (y * z) = x * (z * y) = (x * z) * y = y * (x * z)$, so ($F_2'$) hold. Hence, N(2,0) algebra has a equivalent axiomic system (F_3'), (1) and (2).

Theorem 1.3 Associative *BCI*−algebra is a proper subclass of N(2,0) algebra.

Proof: From reference[2], we know that associative *BCI*−algebra must be N(2,0) algebra, conversely, not ture. For example, let $\mathbf{X}=\{0,1\}$, the operation $*$ is defined as follows:

$*$	0	1
0	0	1
1	1	1

then $(\mathbf{X}, *, 0)$ is a N(2,0) algebra, however, it is not a associtive *BCI*−algebra. It follows that $1*1=1\neq 0$. So N(2,0) algebra is more general than associative *BCI*−algebra.

It is easy to see from [2]:

Theorem 1.4 An N(2,0) algebra $(\mathbf{S}, *, 0)$ become a associative *BCI*−algebra, if the following hold:

(A) $x*x=0$ for any $x\in \mathbf{S}$

or (B) $x*y=y*x=0\Rightarrow x=y$ for any $x,y\in \mathbf{S}$

or (C) $(y*x)*x=y$ for any $x,y\in \mathbf{S}$

Theorem 1.5 Let $(\mathbf{S}, *, 0)$ be a N(2,0) algebra. For any $x,y\in \mathbf{S}$, the following statement are equivalent:

(5) $(x*y)*y=(y*x)*x$

(6) $*$ idempotent

(7) $(x*y)*y=x*y$

Proof: Assume (5); i.e., for every $x,y\in \mathbf{S}$, $(x*y)*y=(y*x)*x$

Let $x=0$, then $(0*y)*y=(y*0)*0$, so $y*y=y$, i.e., $*$ is idempotent, so that (6) hold.

Assume (6); then $(x*y)*y=x*(y*y)=x*y$, so that (7) hold.

Assume (7); i.e., for any $x,y\in S$, $(x*y)*y=x*y$. Let $x=0$, then $y*y=y$, hence $*$ is idempotent, and since $x*y=y*x$, so $x*(y*y)=y*(x*x)$, then

$$(x*y)*y=(y*x)*x,$$

so that (5) hold.

Theorem 1.6 Let $(S,*,0)$ be a N(2,0) algebra, if $*$ is idempotent, then $(S,*,0)$ is a directoid.

Proof: It can be see from reference [3] that a directoid satisfies the following identities for any $x,y,z\in S$,

① $xx=x$, ② $(xy)x=xy$

③ $y(xy)=xy$, ④ $x((xy)z)=(xy)z$

It is easy check that idempotent N(2,0) algebra satisfies ①—④, Hence, an N(2,0) algebra $(S,*,0)$ is a directoid when $*$ is idempotent.

2. Strong N(2,0) algebra

This section we discuss proper subclass of N(2,0) algebra ——Strong N(2,0) algebra.

Definition 2.1 Let $(S,*,0)$ be a N(2,0) algebra. If there exist $x,y\in S$ such that $x*y=y$, then $(S,*,0)$ is called a strong N(2,0) algebra, and we define that $x\leqslant y$ iff $x*y=y$.

Example 2.1 Let $X=\{0,a_1,a_2,\cdots,a_n,1\}$, the binary operation $*$ is defined as follows:

*	0	a_1	a_2	$\cdots$	a_n	1
0	0	a_1	a_2	$\cdots$	a_n	1
a_1	a_1	a_1	a_2	$\cdots$	a_n	1
a_2	a_2	a_2	a_2	$\cdots$	a_n	1
•	•	•	•	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•
a_n	a_n	a_n	a_n	$\cdots$	a_n	1
1	1	1	1	$\cdots$	1	1

It can be shown that $(\mathbf{X}, *, 0)$ is a strong N(2,0) algebra, and

$$0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant 1$$

Example 2.2 Let $\mathbf{X} = \{0,1\}$, the binary operation $*$ is defined as follows:

*	0	1
0	0	1
1	1	0

It is easy to see that $(\mathbf{X}, *, 0)$ is a N(2,0) algebra, but not a strong N(2,0) algebra, because $0 = 1 * 1 \neq 1$.

From the above, we know that strong N(2,0) algebra is a proper subclass of N(2,0) algebra.

Theorem 2.1 Let $(\bar{\mathbf{S}}, *, \leqslant, 0)$ be a strong N(2,0) algebra, then the following statement hold:

1) for any $x \in \bar{S}, 0 \leqslant x$;

2) $*$ is idempotent;

3) $*$ is order—preserving and $a \leqslant a_1, b \leqslant b_1$ implies $a * b \leqslant a_1 * b_1$;

4) $(\bar{S}, \leqslant)$ is a superior—semilattice $(\bar{S}, \vee)$, where $* = \vee$;

5) If $a * x \leqslant b$ has solution on $\bar{S}$, then $a \leqslant b$, and the solution $x \in [0, b]$.

Proof: We only prove 5): Assume $b < a$, i. e., $b * a = a * b = a$, if $x > a$, then $a * x = x \leqslant b$, which is a contradiction to $b < a$;

If $x \leqslant a$, then $a * x = a \leqslant b$, which is a contradiction to $b < a$;

So there are no solution for $b * a \leqslant b$ when $b < a$, hence there exists solution only when $a \leqslant b$, and the solution $x \in [0, b]$.

3. Strong N(2,0) algebra and Lattice implication algebras

In this section, we apply strong N(2,0) algebra to intuitionstic logic.

Let $\bar{S}$ be the set of all propositions, $\leqslant$ is a partical order, which reflects the implication relation between the propositions. $\sim$ is a inverse—order involution operation on $(\bar{S}, \leqslant, *)$, which express the negation. To be simply, we stipulate $\bar{S}$ has a greatest element 1, which express true, as well as a least element 0, which express false, and

$$\sim 0 = 1, \sim 1 = 0, \sim x * x = 1$$

Let $(\bar{S}, *, \leqslant, \sim, 0)$ be a strong N(2,0) algebra, we define:

6) $x \rightarrow y = \sim x * y$

7) $x \bar{o} y = \sim(\sim x * \sim y) = \sim(x \rightarrow \sim y)$

It can be easy to prove that the binary operation $\rightarrow$ and $\bar{o}$ have the following properties:

Theorem 3.1 For any x, y, $z\in\bar{S}$, then

8) $x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$

9) $x\rightarrow(y*z)=(x\rightarrow y)*(x\rightarrow z)$

10) $x\rightarrow(y*z)=y*(x\rightarrow z)$

11) $y\leqslant x*(x\rightarrow y)$

12) $x\,\bar{o}(y\,\bar{o}\,z)=y\,\bar{o}(x\,\bar{o}\,z)$

13) operation $\bar{o}$ is idempotent, associative, isotonic, and commutative, so $(\bar{S},0)$ is a lower semi—latticec and $\wedge=\bar{o}$.

Let $(\bar{S},*,\leqslant,\sim,0)$ is a strong N(2,0) algebra. For any x, $y\in\bar{S}$, if $x\leqslant y$, then $x*y=y$, i. e. , $x\vee y=y$. On the other hand, from $\sim y\leqslant\sim x$, then $x\,\bar{o}y=\sim(\sim x*\sim y)=\sim\sim x=x$, i. e. , $x\wedge y=x$. hence $(\bar{S},*,\bar{o},\leqslant,\sim,0)$ is a lattice. And the order relation "$\leqslant$" is defined by the following:

14) $x\leqslant y$ iff $x\rightarrow y=1$ iff $\sim x*y=1$

Theorem 3.2 Let $(\bar{S},*,\bar{o},\leqslant,\sim,0)$ be a strong N(2,0) algebra, then $a\leqslant b*c$ iff $\sim b\leqslant\sim a*c$, for any $a,b,c\in\bar{S}$.

Proof: since $a\leqslant b*c$ iff $\sim a*(b*c)=1$

iff $\sim b*(\sim a*c)=1$

iff $\sim\sim b*(\sim a*c)=1$

iff $\sim b\leqslant\sim\ a*c$

thus, $a\leqslant b*c$ iff $\sim b\leqslant\sim a*c$.

Theorem 3.3 Let $(\bar{S},*,\bar{o},\leqslant,\sim,0)$ be a N(2,0) algebra, for any $x,y,z\in\bar{S}$, the following statements hold:

15) $\sim(x*y)\leqslant\sim x*\sim y$

16) $(x*y)\rightarrow z\leqslant x\rightarrow(y\rightarrow z)$

17) $x\,\bar{o}(y*z)\leqslant x(y*z)$

18) $x \bar{o} (y \rightarrow z) \leqslant y \rightarrow (x * z)$

19) $x \rightarrow (x \bar{o} y) \leqslant x * (x \rightarrow y)$

20) $(x \bar{o} y) \rightarrow z = x \rightarrow (y \rightarrow z)$

21) $\sim (x \bar{o} y) = \sim x * \sim y, \sim (x * y) = \sim x \bar{o} \sim y$

22) $\sim (x \rightarrow y) = x \bar{o} \sim y$

23) $(x * y) \rightarrow z \leqslant (x \rightarrow z) * (y \rightarrow z)$

24) $(x \bar{o} y) \rightarrow z = (x \rightarrow z) * (y \rightarrow z)$

25) $a \bar{o} b \leqslant c$ iff $a \leqslant b \rightarrow c$ iff $b \leqslant a \rightarrow c$

Proof: We only prove (15) and (24). First since

$x \leqslant 1$ implies $x \leqslant \sim y * y$
implies $x = x * x \leqslant x * (\sim y * y)$
implies $x \leqslant (x * y) * (\sim y)$
implies $\sim (x * y) \leqslant \sim x * \sim y$ (from Th3. 2)

Next, $(x \bar{o} y) \rightarrow z = \sim (x \bar{o} y) * z = (\sim x * \sim y) * z = (\sim x * z) * (\sim y * z) = (x \rightarrow z) * (y \rightarrow z)$.

Theorem 3. 4 Let $(\bar{S}, *, \bar{o}, \leqslant, \sim, 0)$ be a strong N(2,0) algebra. If binary operation $\bar{o}$ and $*$ satisfy the following distributive law: for any $x, y, z \in \bar{S}$,

26) $(x \bar{o} y) * z = (x * z) \bar{o} (y * z)$

then $(\bar{S}, *, \bar{o}, \leqslant, \sim, 0)$ is a lattice implication algebra.

Proof: From the reference [4], for any $x, y, z \in L$, the following identities hold:

(I_1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

(I_2) $x \rightarrow x = 1$

(I_3) $x \rightarrow y = \sim y \rightarrow \sim x$

(I_4) $x \rightarrow y = y \rightarrow x = 1$ implies $x = y$

(I_5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

then $(L, \rightarrow, \sim, 0, 1)$ is a quasi-lattice implication algebra. If it also satisfies:

(L_1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

(L_2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

then $(L, \to, \sim, \vee, \wedge, 0, 1)$ is a lattice implication algebra.

It is easy to check that strong N(2,0) algebra satisfies (I_1)—(I_4). Let $* = \vee$, $\bar{o} = \wedge$. By using 24), we know (L_2) holds, then we only need to prove that (I_5) and (L_1) hold. From the condition identity 26), we have:

$$\begin{aligned}(x \to y) \to y &= \sim(\sim x * y) * y = (x \bar{o} \sim y) * y \\ &= (x * y) \bar{o} (\sim y * y) = (x * y) \bar{o} 1 \\ &= x * y = y * x = (y \to x) \to x \\ (x * y) \to z &= \sim(x * y) * z = (\sim x \bar{o} \sim y) * z \\ &= (\sim x * z) \bar{o} (\sim y * z) \\ &= (x \to z) \bar{o} (y \to z)\end{aligned}$$

Therefore, strong N(2,0) algebra $(\bar{S}, \to, *, \bar{o}, \sim, 0)$ in which the identity 26) holds is a lattice implication algebra.

References

[1] Deng Fang'an, Xu Yang, On N(2,2,0) algebra, Journal of Southwest Jiaotong University, vol. 31, No. 41(1996), 457~463.

[2] Hu Qinping, *BCI*—algebra, Shaanxi science technology Issue (in China) 1987.

[3] Gardner, B. J.; Parmenter, M. M., Directoiods and directed groups. Algebra Universalis 33(95), no. 2, 254~273.

[4] Xu Yang, Lattice Implication Algebra, Journal of Southwest Jiaotong University, vol. 28, No. 1(1993). 20~27.