Fuzzy Homomorphisms of Groups

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Abstract: The concept of fuzzy homomorphism of groups is introduced based on the fuzzy mapping and the fundamental theorem of fuzzy homomorphism is established.

Keywords: Fuzzy mapping; Fuzzy homomorphism; Fuzzy isomorphism.

In this paper G and G' always stand for groups whose unit elements are denoted by e and e', respectively.

Definition 1 Let X and Y be two nonempty sets and let f be a fuzzy subset of $X*Y$. If for $\forall x \in X$, there exits a unique $y_x \in Y$ such that $f(x, y_x) = 1$, then f is called a fuzzy mapping from X to Y.

We denote $f: X \cdots \rightarrow Y$ in case f is a fuzzy mapping from X to Y. We call f onto if $\forall y \in Y, \exists x \in X$, such that $f(x, y) = 1$. The fuzzy mapping f is one-one if $f(x_1, y) = f(x_2, y) = 1 \quad \Rightarrow \quad x_1 = x_2$.

Definition 2 A fuzzy mapping $f: G \cdots \rightarrow G'$ is called a fuzzy homomorphism if for $\forall x_1, x_2 \in G, \forall y \in G'$,

$f(x_1 x_2, y) = \sup\{f(x_1, y_1) \wedge f(x_2, y_2) \mid y = y_1 y_2\}$.

If a fuzzy homomorphism $f: G \cdots \rightarrow G'$ is onto, we write $G \sim G'$. A fuzzy homomorphism $f: G \cdots \to G'$ is called a fuzzy isomorphism if f is both one-one and onto. In this case, we write $G \cong G'$.

Proposition 3 Let $f: G \cdots \to G'$ be a fuzzy homomorphism. Then

(1) $f(e, e') = 1$ (2) $f(x, y) = 1 \Rightarrow f(x^{-1}, y^{-1}) = 1$

(3) $(y_x)^{-1} = y_{x^{-1}}$, $\forall x \in G$ (4) $y_{x_1 x_2} = y_{x_1} y_{x_2}$, $\forall x_1, x_2 \in G$.

Proposition 4 Let $f: G \cdots \to G'$ be an onto fuzzy homomorphism. Then $N = \{x \in G \mid f(x, e') = 1\}$ is a normal subgroup of G and G/N is isomorphic to G'.

Proposition 5 Let $f: G \cdots \to G'$ be a fuzzy homomorphism and let A be a fuzzy subgroup of G. Then $f(A)$ is a fuzzy subgroup of G', where $f(A)(y) = \sup\{A(x) \mid f(x, y) = 1\}, \forall y \in G'$. Moreover, if A is a fuzzy normal subgroup of G and f is onto, then $f(A)$ is a fuzzy normal subgroup of G'.

Proof: For $\forall y_1, y_2, y \in G'$, we have

$$f(A)(y_1 y_2) = \sup\{A(x_1 x_2) \mid f(x_1 x_2, y_1 y_2) = 1\} \geq \sup\{A(x_1) \wedge A(x_2) \mid f(x_1, y_1) = 1, f(x_2, y_2) = 1\}$$

$$= \sup\{A(x_1) \mid f(x_1, y_1) = 1\} \wedge \sup\{A(x_2) \mid f(x_2, y_2) = 1\} = f(A)(y_1) \wedge f(A)(y_2),$$

$$f(A)(y^{-1}) = \sup\{A(x^{-1}) \mid f(x^{-1}, y^{-1}) = 1\} = \sup\{A(x) \mid f(x, y) = 1\} = f(A)(y).$$

So, $f(A)$ is a fuzzy subgroup of G'.

If A is a fuzzy normal subgroup of G and f is onto, then there exists $x_1 \in G$, such that $f(x_1, y_1) = 1$. So,

$$f(A)(y_1^{-1} y y_1) = \sup\{A(x_1^{-1} x x_1) \mid f(x_1^{-1} x x_1, y_1^{-1} y y_1) = 1\}$$

$$\geq \sup\{A(x) \mid f(x, y) = 1\} = f(A)(y)$$

That is, $f(A)$ is a fuzzy normal subgroup of G'.

Proposition 6 Let $f: G \cdots \to G'$ be a fuzzy homomorphismand and let *B* be a fuzzy subgroup (fuzzy normal subgroup) of G'. Then $f^{-1}(B)$ is a fuzzy subgroup (fuzzy normal subgroup) of *G*, where $f^{-1}(B)(x) = B(y_x), \forall x \in G$.

Proof: For $\forall x_1, x_2 \in G$, we have $f^{-1}(B)(x_1^{-1}x_2) = B(y_{x_1^{-1}x_2}) = B((y_{x_1})^{-1} y_{x_2})$

$$\geq B(y_{x_1}) \wedge B(y_{x_2}) = f^{-1}(B)(x_1) \wedge f^{-1}(B)(x_2).$$

So, $f^{-1}(B)$ is a fuzzy subgroup of *G*.

If *B* is a fuzzy normal subgroup, Then

$$f^{-1}(B)(x_1^{-1}x_2x_1) = B(y_{x_1^{-1}x_2x_1}) = B((y_{x_1})^{-1} y_{x_2} y_{x_1}) = B(y_{x_2}) = f^{-1}(B)(x_2)$$

Hence, $f^{-1}(B)$ is a fuzzy normal subgroup of *G*.

Proposition 7 Let *A* be a fuzzy normal subgroup of *G* and *A*(*e*)=1. Then $G \sim G/A$.

Proof: Let $f(x_1, x_2A) = A(x_1^{-1}x_2), \forall x_1, x_2 \in G$. Now we show that $f: G \to G'$ is an onto fuzzy homomorphism.

For $\forall x \in G$, we have $f(x, xA) = A(x^{-1}x) = A(e) = 1$. If $\exists\ x_1, x_2 \in G$, such that $f(x, x_1A) = f(x, x_2A) = 1$, then $A(x^{-1}x_1) = A(x^{-1}x_2) = 1$. So $A(x_1^{-1}x_2) = A(x_1^{-1}xx^{-1}x_2) \geq A(x_1^{-1}x) \wedge A(x^{-1}x_2) = 1$. i.e. $A(x_1^{-1}x_2) = 1$, $x_1A = x_2A$. This show that $f: G \to G/A$ is an onto fuzzy mapping. Moreover, if for $\forall x_1, x_2, y \in G$, $\exists\ y_1, y_2 \in G$, such that $yA = (y_1A)(y_2A)$, then

$$f(x_1x_2, yA) = f(x_1x_2, (y_1A)(y_2A)) = f(x_1x_2, y_1y_2A) = A(x_2^{-1}x_1^{-1}y_1y_2)$$

$$\geq A(x_2^{-1}y_2) \wedge A(y_2^{-1}x_1^{-1}y_1y_2) = A(x_1^{-1}y_1) \wedge A(x_2^{-1}y_2) = f(x_1, y_1A) \wedge f(x_2, y_2A).$$

So, $f(x_1x_2, yA) \geq \sup\{f(x_1, y_1A) \wedge f(x_2, y_2A) \mid yA = (y_1A)(y_2A)\}$,

$$\sup\{f(x_1, y_1A) \wedge f(x_2, y_2A) \mid yA = (y_1A)(y_2A)\} \geq f(x_1, x_1A) \wedge f(x_2, x_1^{-1}yA)$$

$$= f(x_2, x_1^{-1}yA) = A(x_2^{-1}x_1^{-1}y) = f(x_1x_2, yA)$$

Hence, $f(x_1x_2, yA) = \sup\{f(x_1, y_1A) \wedge f(x_2, y_2A) \mid yA = (y_1A)(y_2A)\}$.

That is, $G \sim G/A$.

Proposition 8 Let $f: G \cdots \to G'$ be an onto fuzzy homomorphism. Then

(1) A is a fuzzy normal subgroup of *G*, where

$$A(x) = f(x, e') \wedge f(x^{-1}, e'), \quad \forall x \in G.$$

(2) $G/A \cong G'$.

Proof: (1) For $\forall x, a \in G$, we have

$$A(xa) = f(xa, e') \wedge f((xa)^{-1}, e') \geq f(x, e') \wedge f(a, e') \wedge f(a^{-1}, e') \wedge f(x^{-1}, e')$$

$$= [f(x, e') \wedge f(x^{-1}, e')] \wedge [f(a, e') \wedge f(a^{-1}, e')] = A(x) \wedge A(a),$$

$A(x^{-1}) = A(x)$. So A is a fuzzy subgroup of *G*. Further,

$$A(a^{-1}xa) = f(a^{-1}xa, e') \wedge f(a^{-1}x^{-1}a, e') = f(a^{-1}xa, y_a^{-1}e'y_a) \wedge f(a^{-1}x^{-1}a, y_a^{-1}e'y_a)$$

$$\geq f(a^{-1}, y_a^{-1}) \wedge f(x, e') \wedge f(a, y_a) \wedge f(a^{-1}, y_a^{-1}) \wedge f(x^{-1}, e') \wedge f(a, y_a)$$

$$= f(x, e') \wedge f(x^{-1}, e') = A(x).$$

Hence, A is a fuzzy normal subgroup of *G*.

(2) Let $h(xA, y) = f(x, y), \forall x \in G, \forall y \in G'$. Then there exists $y_x \in G'$, such that $h(xA, y_x) = f(x, y_x) = 1$. If $x_1A = x_2A$ and $\exists\ y_1, y_2 \in G'$ such that $h(x_1A, y_1) = h(x_2A, y_2) = 1$, then $A(x_1^{-1}x_2) = A(e) = 1$.

So, $f(x_1^{-1}x_2, e') \wedge f(x_2^{-1}x_1, e') = 1$, $f(x_1^{-1}x_2, e') = 1$.

From $f(x_1^{-1}x_2, y_1^{-1}y_2) \geq f(x_1^{-1}, y_1^{-1}) \wedge f(x_2, y_2) = 1$, we can obtain that $y_1^{-1}y_2 = e'$ and $y_1 = y_2$. Hence, $h: G/A \cdots \to G'$ is an onto fuzzy mapping.

If $\exists\ x_1, x_2 \in G$, such that $h(x_1A, y) = h(x_2A, y) = 1$, then $f(x_1, y) = f(x_2, y) = 1$. So, $f(x_1^{-1}x_2, e') \geq f(x_1^{-1}, y^{-1}) \wedge f(x_2, y) = 1$ and hence $A(x_1^{-1}x_2) = 1 = A(e)$. i.e. $x_1A = x_2A$. It shows that *h* is one-one.

For $\forall x_1, x_2 \in G, \forall y \in G'$, we have

$$h((x_1A)(x_2A), y) = h((x_1x_2A), y) = f(x_1x_2, y) = \sup\{f(x_1, y_1) \wedge f(x_2, y_2) \mid y = y_1y_2\}$$

$$= \sup\{h(x_1A, y_1) \wedge f(x_2A, y_2) \mid y = y_1y_2\}.$$

So, $h: G/A \cdots \to G'$ is a fuzzy isomorphism and hence $G/A \cong G'$.

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