# On Regular Representations of Hypergrorups①

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**Abstract** In this paper, the operational properties of the hypergroups on a finite group are discussed. By the left translation action of the hypergroup, the regular representations and the matrix representations of a hypergroup on a finite group are given.

**Keyword** Hypergroup, Group action, regular representation, Matrix representation

The theoretical needs of the set - value mapping lead the upgrade of some mathematical constructures. Prof. Li Hongxing etc[1,2,3] first intrduced the concept of hypergroup which creat the study of hyperalgebra, moreover, some useful results are obtained. Because the operations in a hyperalgebra is based on the operations of some elements in the base algebra, it is worth to study how to represent directly these operations and to judge whether a subset of the powerset is a certain algebric constructure. In this paper, the operational properties of the hypergroups on a finite group are discussed. By the left translation action of the hypergroup, the regular representations and the matrix representations of a hypergroup on a finite group are given.

## 1. Introduction

Let G be an arbitrary group and P(G), the powerset of G. Under the subset multiplication

$$AB \triangleq \{ab \mid a \in A,\ b \in B\},$$

$P_o(G) = P(G) - \{\varnothing\}$ forms a semigroup which have the identity. A subgroup $\mathscr{G}$ of $P_o(G)$ is called a hypergroup on G, and G, the generating group of $\mathscr{G}$ . the identity of $\mathscr{G}$ is denoted by E and the $G^* \triangleq \bigcup\{A \mid A \in \mathscr{G}\}$ is called the basic elements set.

Let $\mathscr{G}$ be a hypergroup on G , we have following conclusions:

**Lemma 1.1**[2] $\forall A \in \mathscr{G}$ , $|A| = |E|$. ($|A|$ is the base number of A)

**Lemma 1.2**[2] $\forall A, B \in \mathscr{G}$ , $A \cap B \neq \varnothing \Rightarrow |A \cap B| = |E|$.

**Lemma 1.3**[5] Let $\mathscr{G}$ be a hypergroup on a finite group G, then

(1) $E \leqslant G$;

(2) $G^* \leqslant G$;

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(3) $\forall A \in \mathscr{G}$, $\forall a \in A, aE = Ea = A$, i.e, $E \lhd G^*$;

(4) If $aE = bE$, then $a, b \in A \in \mathscr{G}$;

(5) $\forall A \in \mathscr{G}$, $t \in E$ iff $tA = At = A$;

(6) $\forall A \in \mathscr{G}$, If $aE = A$, then $a^{-1}E = A^{-1}$;

(7) $\forall A, B \in \mathscr{G}$, If $AB = C$, then $\forall a \in A$, $aB = C$;

(8) $\forall B \in \mathscr{G}$, $\forall a \in A \in \mathscr{G}$, $xB = aB$ iff $x \in A$;

(9) $\mathscr{G} = G^*/E$.

## 2. Regular Representations of Hypergroups

**Definition 2.1** Let G be a group and X a non-empty set. A action of G on a set X is a map from $G \times X$ to X, with the image of $(g, x)$ being denoted by $g(x)$, which satisfies the following conditions:

(1) $e(x) = x$, for every $x \in X$,

(2) $g_1g_2(x) = g_1(g_2(x))$, for every $g_1, g_2 \in G$ and $x \in X$,

Let G be a finite group and $\mathscr{G}$, a hypergroup on G, then we have an action of $\mathscr{G}$ on $\mathscr{G} = \{A_1, A_2, \cdots, A_m\}$: $\mathscr{G} \times \mathscr{G} \to\to \mathscr{G}$, $(T, A_i) \to TA_i$ for every $T, A_i \in \mathscr{G}$ $(i = 1, 2 \cdots m)$, it is called the regular translation ation of $\mathscr{G}$, where $m = |\mathscr{G}|$.

Let K be a field and

$V \triangleq \{k_1A_1 + \cdots + k_mA_m \mid k_i \in K, i = 1, 2 \cdots m\}$

defining

$\sum_{i=1}^{m} k_iA_i = \sum_{i=1}^{m} l_iA_i$ iff $K_i = l_i$, $i = 1, 2 \cdots m$;

$$\sum_{i=1}^{m} k_iA_i + \sum_{i=1}^{m} l_iA_i \triangleq \sum_{i=1}^{m} (k_il_i)A_i$$

$$k(\sum_{i=1}^{m} k_iA_i) \triangleq \sum_{i=1}^{m} (kk_i)A_i$$

for every $k, k_i, l_i \in K$.

Then V is a m-dimension vector space on K. And $\{A_1, A_2, \cdots, A_m\}$ is a basis of vector space V.

For $A \in \mathscr{G}$, defining

$$\varphi(A)(\sum_{i=1}^{m} k_iA_i) \triangleq \sum_{i=1}^{m} k_i(AA_i)$$

then $\varphi(A)$ is a linear transformation of V.

If $i \neq j$ and $AA_i = AA_j$, then $A_i = A_j$, this is a contradicfion. Thus $\{AA_1, AA_2, \cdots, AA_m\}$ is also a basis of V and $\varphi$ (A) is a invertible linear transformation.

And $\varphi: \mathcal{G} \to GL(V)$, $A \to \varphi(A)$

is a map which satisfies condition $\varphi(AB) = \varphi(A)\varphi(B)$ for every $A, B \in \mathcal{G}$.

Hence $\varphi$ is a linear representation of the hypergroup $\mathcal{G}$ on K, which is called the regular representation of $\mathcal{G}$.

For $\{A_1, A_2, \cdots, A_m\}$, the basis of V, $\varphi$ defines a matrix representation $\Phi$ such that for every $A \in \mathcal{G}$, $\Phi(A)$ is a permutation matrix in which every row and column has a unique non-zero entry and all non-zero entries are equal to 1.

**Example 2.1** Let $G = \langle a \rangle$ be a cyclic group and $a^{12} = e$, $\mathcal{G} = \{A_1, A_2, A_3\}$ where $A_1 = \{e, a^6\}$, $A_2 = \{a^2, a^8\}$, $A_3 = \{a^4, a^{10}\}$. then $\mathcal{G}$ is a hypergroup on G, and $G^* = \{e, a^2, a^4, a^6, a^8, a^{10}\}$.

Let $V \triangleq \{k_1A_1 + k_2A_2 + k_3A_3 \mid k_i \in K\}$, then $\{A_1, A_2, A_3\}$ is a basis of V.

and $$\Phi(A_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Phi(A_2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\Phi(A_3) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. .$$

**Proposition2.1** $\Phi(A)$ is the orthogonal matrix, for every $A \in \mathcal{G}$ .

Proof. since $\Phi(A)$ is a permutation matrix, then $\Phi(A) = (e_{i_1}, e_{i_2}, \cdots, e_{i_m})$ Where $i_1, i_2, \cdots, i_m$ is a permutation of $1, 2, \cdots, n$ and $e_i$ is the m-dimension standard identity vector, i. e.

$$e_i = (0, \cdots 0, \underset{ith}{1}, 0, \cdots 0)^T \quad 1 \leqslant i \leqslant m$$

thus, $\Phi(A)^T \cdot \Phi(A) = I_m$, $\Phi(A)$ is an orthogonal matrix.

**Proposition2.2** $(\Phi(A))^{-1} = \Phi(A^{-1})$.

**Proposition2.3** For every $A, B \in \mathcal{G}$, $\Phi(A) \cdot \Phi(B) = \Phi(AB)$.

These Proof are strainghtforward.

**Theorem2.1** Let $\mathcal{G}$ be a hypergroup on G and $\Phi$, the regular matrix represmtation of $\mathcal{G}$,

$$\Phi(\mathscr{G}) \triangleq \{\Phi(A) | A \in \mathscr{G}\},$$

then $\Phi(\mathscr{G})$ is a group and $\Phi(\mathscr{G}) \cong \mathscr{G}$.

**Proof.** By proposition2.1, 2.2, 2.3, $\Phi(\mathscr{G})$ forms a group under the matrix multiplication. Let $f: \mathscr{G} \to\to \Phi(\mathscr{G})$, $A \to \Phi(A)$, then f is an epimorphism.

If $\Phi(A) = \Phi(B)$, then $\varphi(A) = \varphi(B)$, for arbitrary $T \in \mathscr{G}$, $AT = BT$ implies $A = B$, hence f is bijective.

For every A, $B \in \mathscr{G}$, $f(AB) = \Phi(AB) = \Phi(A)\Phi(B) = f(A)f(B)$

Hence $\Phi(\mathscr{G}) \cong \mathscr{G}$.

**Corollary2.1** $\mathscr{G}$ is abelian iff $\Phi(\mathscr{G})$ so is.

**Corollary2.2** Let $\mathscr{L}(\Phi(\mathscr{G}))$ be a vector space on K generated by $\Phi(\mathscr{G})$, then

$$\mathscr{L}(\Phi(\mathscr{G})) \cong V.$$

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**References**

1. Li Hongxing, Wang Peizhuang. Hypergroup. BUSFAL(23)1985, 22 - 29.

2. Li Hongxing, Wang Peizhuang. Hypergroup. Mathematics Applicata, (1)1988, 1 - 4.

3. W. Yang, Groups where every elements is noempty subset of a group, J. Southwest Teach. Col. (2) 1985, 106 - 108.

4. He Qing, Li Hongxing, Representations of Power groups and isomorphic upgrade of power groups, Journal of Beijing Normal university(Natural Science) (1)1999, 32 - 37

5. Zhang Chengyi, Zheng Qingan, The operational property of a hypergroup, Academic forum of Nan Du, (to apear.)

6. J. L. Alperin and R. B. Bell, Group and Representations, Springer - verlag, 1995,

7. Chao Xihua, Ye jiachen, Theory of Group representations, Beijing University, 1998.