Fuzzy module over fuzzy algebra

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Abstract:

In this paper, we will introduce the concepts of fuzzy modules over fuzzy algebra and discuss the important properties of it.

Keywords:

algebra, algebra module, fuzzy set, level subset, lattice, complete lattice, homomorphism, fuzzy module over fuzzy alyebra.

1. Fuzzy modules over fuzzy algebra and its operations.

Let X be any set and L a boundecl lattice with 1 and 0, then a fuzzy set V_x in X is charalterized by a mapping $V_x: X \to L$. The set

$$V_{X_\alpha} = \{x \in X \mid V_X(x) \geqslant \alpha\}$$

is called a level subset of X with respect to V_x, where $\alpha \in L$.

Unlesss specially stated, M in this article only refers to the left module ove algebra A, in brief, M is an A - module, where A is an algebra over field F.

Definition1.1. Let A be an algebra over field F, then fuzzy subset V_A of A is called a fuzzy algebra of if for all $a_1, a_2, \in A$, and $\alpha \in F$, we have

1) $V_A(a_1 - a_2) \geqslant V_A(a_1) \wedge V_A(a_2)$,

2) $V_A(a_1 a_2) \geqslant V_A(a_1) \wedge A(a_2)$,

)3 $V_A(\alpha a_1) \geqslant V_A(a_1)$,

4) $V_A(0) = 1$.

Definition1.2. The fuzzy subset V_M of M is called module over fuzzy alyebra A, if for all $m_1, m_2 \in M$, $a \in A$, We have

1) $V_M(x - y) \geqslant V_M(x) \wedge V_M(y)$,

2) $V_M(0) = 1$,

3) $V_M(ax) \geqslant V_A(a) \wedge V_M(x)$.

In brief V_M is a V_A - fuzzy module.

Definition 1.3. Let M a A - module, C_M, B_M are fuzzy subset of M, $\alpha \in A$, definition fuzzy subset of M: $C_M \cap B_M$, $C_M + B_M$, aC_M, $-C_M$, for $x \in M$, they make

$$(C_M \cap B_M)(x) = C_M(x) \wedge B_M(x),$$
$$(C_M + B_M)(x) = \bigvee_{x_1+x_2=x} [C_M(x_1) \wedge B_M(x_2)],$$

$$(aC_M)(x) = \bigvee_{ax_1=x} C_M(x_2)$$
$$(-C_M)(x) = C_M(-x).$$

Similar to Theorem1.1 − 1.3, We can easily prove the following Theorem 1.1 − 1.3.

Theorem1.1 Let V_A be a fuzzy algebra of algebra A and V_M be a fuzzy subset of A − module M, then V_M is a V_A − fuzzy module iff all $\alpha \in L$, V_{M_α} is alyebra module over aclgebra V_{A_α}.

Theorem 1.2. Let L be a complete lattice, the fuzzy subset V_M of M is a V_A − fuzzy module iff there is a subalgebra family

$$\{ Y_\alpha \mid \alpha \in L, Y_\alpha \textit{ is a subalgebra of } A, \bigcap_{\substack{\alpha \in H \\ H \subseteq L}} Y_\alpha \subseteq Y_{suppH} \}$$

and subset family

$$\{ X_\alpha \mid \alpha \in L, X_\alpha \subseteq M, \bigcap_{\substack{\alpha \in H \\ H \subseteq L}} X_\alpha \subseteq X_{suppH} \},$$

this makes X_α that is an additive subgroup, in particular if $\alpha \leqslant \sup_{a \in A} V_A(a)$, then X_α is an A − module, *and* $V_A = \bigcup_{\alpha \in L} \alpha . \cdot \tilde{Y}_\alpha$ $V_M = \bigcup_{\alpha \in L} \alpha \cdot \tilde{X}_\alpha$, *here* $\tilde{Y}_\alpha$ *and* $\tilde{X}_\alpha$ respectively indicate characteristic functions of Y_α and X_α.

Wheorem1.3. Let L be a complete lattice, then $\bigcap_{k \in I} V_M^{(k)}$ is a V_A − fuzzy submodule, here $\{V_M^{(k)}\}_{k \in I}$ is a family of V_A − fuzzy submodule. if L is a fimite set, then for any $\alpha \in L$, $(\bigcap_{k \in I} V_M^{(k)})_\alpha = \bigcap_{k \in I} V_{M_\alpha}^{(k)}$

Proposition1.4. Let C is a fuzzy set of X, for all $x \in X$, let

$$J_x = \{ \alpha \mid \alpha \in C_\alpha \}$$

then J_x is the ideal of L which is generatecl by c(x), and $c(x) = \sup J_x$.

Proposition 1.5. Let V_M be a V_A − module of M, then for all $x, y \in M$, $a \in A$, $J_{-x} \geqslant J_x$, $J_{x+y} \geqslant J_x \cap J_y$, $J_{ax} \geqslant J_x \cap J_a$, where J_x, J_{x+y}, J_{-x}, J_a, J_{ax} are ideals of l which is defined by Proposition1.4.

Proof. The proof is sinilar to Proposition1.6 of [3].

Theorem1.6. Let V_M is fuzzy set of M, then V_M is a V_A − module iff

(1) $\bigvee_{x \in M} B_M(x) = 1$

(2) $J_{-x} \geqslant J_x, J_{x+y} \geqslant J_x \cap J_y, J_{ax} \geqslant J_x \cap J_a$, *for all* $x, y \in M, a \in R$.

Proof. If V_M is a V_A − module, it is easy to prove (1) and (2). Coversely, we prove that V_M is an V_A − fuzzy module.

In virtue of Proposition1.4 $V_M(x) = \sup J_x$, so

$V_M(x+y) \geqslant \sup J_{x+y} \geqslant \sup < V_M(x) \wedge V_M(y) > V_M(x) \wedge V_M(y)$,

$$V_M(-x) \geqslant supJ_{-x} \geqslant supJ_x = V_M(x),$$

$$V_M(ax) \geqslant supJ_{ax} \geqslant supJ_a \wedge supJ_x = V_A(a) \wedge V_M(x),$$

for al x、$y \in M$, $\alpha \in A$

For all $x \in M$, $0 = 0.x$, thas $V_M(0) = V_M(0.x) \geqslant V_M(x)$, hence $V_M(0) \geqslant \bigvee_{x \in M} V_M(x) = 1$, there fore V_M is a V_A – fuzzy module

Theorem1.7 Let C_M, B_M are V_A – fuzzy module of M, then $C_M + B_M$ is a V_A – fuzzy module of M.

Theorem 1.7 can be easily drawn.

2 Relation of fuzzy submodules betueen homomorphism mrodules.

Let M and N be two A – module, f is a homomorphism mapping from M to N, V_A is a fvzzy algebra of A. S(M) and S(N) stands for the set composed of all the V_M – fuzzy module of M and N respectively. Let $V_M \in S(M)$, them $f(V_M)$ is defined by:

$$f(V_M)(x') \geqslant \begin{cases} \vee \ \{V_M(x) \mid x \in f^{-1}(x'), f^{-1} \neq (x') \neq \phi\} \\ 0 \ \ if\ f^{-1}(x) = \phi \end{cases}$$

for all $x' \in N$. Let $V_N \in S(N)$, then $f^{-1}(V_N)$ is defined by:

$$f^{-1}(V_M)(x) = V_N(f(x))$$

for all $x \in M$.

Lemma 2.1. Let $V_M, V_M^1, V_M^2 \in S(M)$, $V_N, V_N^1, V_N^2 \in S(M')$, then

(1) $f(V_N)(0') = 1$, where $0'$ stands for zero of N,

(2) $f(f^{-1}(V_N)) = V_N$,

(3) If $V_M^1 \subseteq V_M^2$, then $f(V_M^1) \subseteq f(V_m^2)$,

(4) If $V_N^1 \subseteq V_N^2$, then $f^{-1}(V_N^1) \subseteq f^{-1}(V_N^2)$,

(5) $f(V_M^1 \cap V_M^2) = f(V_M^1) \cap f(V_M^2))$,

(6) If V_M^1 and V_M^2 are all constant 1 on Kerf, then $f(V_M^1 + V_M^2) = f(V_M^1) + f(V_M^2)$,

(7) If $V_N = f(V_M)$, V_M is constant 1 on Kerf , then $V_M = f^{-1}(V_N)$,

(8) $f(V_{M_o}) \subseteq (f(V_M)_o$

(9) If V_M is constant 1 on kerf, then for and $x \in M$, we have $f(V_M)(f(x)) = V_M(x)$.

Theorem 2.1. If $V_M \in S(M)$, $V_N \in S(N)$, then

(1)$f^{-1}(V_N)$ is a V_A − fuzzy module of M and constant 1 on kerf,

(2)$f^{-1}(V_{N_0}) = (f^{-1}(V_N))_o$,

(3)If V_M is constant 1 on kerf, then $f^{-1}(f(V_M)) = V_M$.

Lemma 2.2 Let L be a complete lattice, M and N are two A − modules, $f: M \to N$ is an epimorphism, then if V_M is a V_A − fuzzy module of M, then $f(V_M)$ is a V_A − fuzzy module, and furthermore , if V_N is constant 1 on kerf, then $f(V_{M_0}) = (f(V_M))_0$.

Proof Because V_M is V_A − fuzzy module, then for any $x', y' \in N$ and $a \in A$,

$$f(V_M)(x'-y') = \{V_M(z) \mid z \in f^{-1}(x'-y')$$
$$\geqslant \vee \{V_M(x-y) \mid x \in f^{-1}(x'), y \in f^{-1}(y')$$
$$= (\vee \{V_m(x) \mid x \in f^{-1}(x') \wedge (\vee \{V_M(y) \mid y \in f^{-1}(y')$$
$$= f(V_M)(x') \wedge f(V_m)(y'),$$

$$f(V_M)(ax') = \vee \{V_M(z) \mid z \in f^{-1}(ax')\} \geqslant \vee \{V_M(ax) \mid x \in f^{-1}(x')\}$$
$$\geqslant \vee \{V_M(x) \wedge V_A(a) \mid x \in f^{-1}(x')\}$$
$$= V_A(a) \wedge (\vee \{V_M(x) \mid x \in f^{-1}(x')\}$$
$$= V_A(a) \wedge (\vee \{V_M(x) \mid x \in f^{-1}(x')\}$$
$$= V_A(a) \wedge f(V_M)(x')$$

By **lemma** 2.1. We have $f(V_M)(0') = 1$, consequently $f(V_M)$ is a V_R − fuzzy module.

If $x' \in (f(V_M))_o$, then $f(V_M)(x') = 1$ because f is epinorphism, so $\exists x \in M$, it makes $f(x) = x'$. According to Lemma2.1 we get $f(V_M)(x') = f(V_M)$, $f(x) = V_M(x) = 1$, thas $x' = f(x)$, $x \in V_{M_o}$, and $x' \in f(V_M)$, ie, $(f^{-1}(V_M))_0 \subseteq f(V_M)_0$, therefore$(f(V_M)_0 = f(V_{M0})$

Theorem 2.2 let M and N be two left modules over ring K, $f: M \to N'$is am epimorphism and L is a complete distributive lattice , then there is a one − to − one order preserving correspondence between the V_A − fuzzy modules of M and these of N which are constant 1 on Kerf.

Proof Let K(M) be the set of all V_A − fuzzy module of M which are constant 1 on Kerf. Let $\varphi: K(M) \to S(N)$ and $\varphi: S(N) \to K(M)$ be defined as $\varphi(V_M) = f(V_M)$ and $\varphi(V_N) = f^{-1}(V_N)$.

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