Solution of E.Sanchez's Fuzzy Relation Equations based on the t-norm

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Abstract. In this paper we consider some results related to find the solution the Sup-(t-norm) fuzzy relations equations which based on the additive generator of the t-norm.

Keywords: Fuzzy relation, t-norm, additive generator, α-operator.

Introduction. The concept of fuzzy relation equations was introduced by E.Sanchez [2]. In that paper it was represented a metod for the solution of the Sup-min fuzzy relation equations that give the greatest element of the sets of solution, if it exists, and from which many applications can be done.

The purpose this work is to generalise that results and to find the solution of the Sup-(t-norm) fuzzy relation equations.

Our method is based on the function representation of E.Sanchez's α-operator.

α-operator: $[0;1] \times [0;1] \to [0;1]$ is an operator associated with the t-norm $T(x,y) = \min(x,y)$ and defined as [2, 3]:

$$\mathbf{x}\,\alpha\,\mathbf{y} = \mathrm{Sup}(z \in [0;1]:\ T(x,z) \le y) = \begin{cases} 1, \text{ if } x \le y, \\ 0, \text{ if } x>y. \end{cases}$$

$\mathbf{x}\,\alpha\,\mathbf{y}$ is the relative pseudocomlpement of **x** in **y**.

A triangular norm [1] (briefly: t-norm) is a function of the two variables T: $[0;1] \times [0;1] \to [0;1]$, such that $T(x,0)=0$; $T(x,1)=1$; $T(x,y)=T(y,x)$; $T(x,y)\le T(y,x)$, if $y\le z$; $T(x, T(y,z))=T(T(x,y), z)$, where $x, y, z \in [0;1]$.

If T is an Archimedian t-norm, then there exist a continuous and decreasing function **f**: $[0;1] \to [0;\infty]$, $\mathbf{f}(1)=0$, such that T is representable in the form

$$T(x,y) = f^{-1}(\min(f(0), f(x) + f(y))),$$ where $\mathbf{f}^{-1}$ is the inverse of **f**.

Several examples of t-norms and corresponding α-operators we are listed below:

Example 1. Well known E.Sanchez's operator

$$\mathbf{x}\,\alpha\,\mathbf{y} = \begin{cases} 1, \text{ if } x \le y \\ 0, \text{ if } x>y \end{cases} \quad \text{with } T(x,y) = \min(x,y).$$

Example 2. For $T(x,y) = xy$, we have $$\mathbf{x}\,\alpha\,\mathbf{y} = \begin{cases} 1, \text{ if } x \le y, \\ y/x, \text{ if } x>y. \end{cases}$$

Example 3. Let $T(x,y)= \max(0, x+y-1)$, then $\mathbf{x}\,\alpha\,\mathbf{y} = \min(1, 1+y-x)$.

Example 4. If

$$T(x,y)=\begin{cases} x, & \text{if } y=1, \\ y, & \text{if } x=1, \\ 0, & \text{else} \end{cases} \quad \text{then} \quad \mathbf{x}\,\alpha\,\mathbf{y}=\begin{cases} 1, & \text{if } x \le y, \\ 1, & \text{if } 1>x>y, \\ y, & \text{if } 1=y>x. \end{cases}$$

The following properties have been proved in [2]:

$\mathbf{x} \wedge (\mathbf{x}\,\alpha\,\mathbf{y}) \le y$; $\mathbf{x}\,\alpha\,(\mathbf{y} \vee \mathbf{z}) \ge \mathbf{x}\,\alpha\,\mathbf{y}$; $\mathbf{x}\,\alpha\,(\mathbf{x} \wedge \mathbf{y}) \ge \mathbf{y}$.

Results. For the determination of solution our problems we will formulate the representation theorem for α-operator.

Theorem 1. If $T(x,y) = f^{-1}(\min(f(0), f(x) + f(y)))$ then correspondingly α-operator is

$$\mathbf{x}\,\alpha\,\mathbf{y} = \mathrm{Sup}(z \in [0;1]:\ T(x,z) \le y) = f^{-1}(\max(0,\ f(y) - f(x))).$$

Proof. Let $f(z) = \max(f(1), f(y) - f(x))$.

If $x>y$, then $f(z) = f(y) - f(x)$ and $z = f^{-1}(f(y) - f(x))$

If $x \le y$, then $f(z) = f(1)$ and $z = f^{-1}(f(1)) = f^{-1}(0)$.

Note 1. The α-operator $\mathbf{x}\,\alpha\,\mathbf{0} = f^{-1}(f(0) - f(x))$ is well known negation (function) which is denoted by $n(x)$ [1], [4].

Note 2. The α-operator $\mathbf{x}\,\alpha\,\mathbf{y} = f^{-1}(f(y) - f(x))$, for $x>y$ is well known implication which denoted by $(x\Rightarrow y)$ [1], [4].

Example 5. Let $f(z)=(1-z)^p$, $p>0$. In this case we have Yager's operator $T(x,y) = 1 - \min(1, ((1-x)^p + (1-y)^p)^{1/p})$ and in accordance with Theorem 1

$$\mathbf{x}\,\alpha\,\mathbf{y} = 1 - \max(0, (1-y)^p + (1-x)^p)^{1/p}$$

It is easy to verify that from Example 5, when $p \to \infty$ we get Example 1; if $p=1$, then we have Example 3; and when $p \to 0$, then we get Example 4.

Theorem 2. For $x, y, z \in [0;1]$ we have

$$T(x,\ \mathbf{x}\,\alpha\,\mathbf{y}) \le y \qquad (1)$$

$$\mathbf{x}\,\alpha\,\mathbf{max(y, z)} \ge \mathbf{x}\,\alpha\,\mathbf{y} \text{ or } (\mathbf{x}\,\alpha\,\mathbf{z}) \qquad (2)$$

$$\mathbf{x}\,\alpha\,\mathbf{T(x, y)} \ge y \qquad (3)$$

Proof. Propositions 1,2,3 of the Theorem 2 are the analogue of Sanchez's inequalities, but we proved its use in the Theorem 1 as well.

1. Proposition 1 is a consequence of the definition α- operator.

2. Obviously, $\mathbf{x}\,\alpha\,\mathbf{max(y, z)} = \max(\mathbf{x}\,\alpha\,\mathbf{y},\ \mathbf{x}\,\alpha\,\mathbf{z})$. From this, the proof of the proposition 2 follows.

3. According to Theorem 1 we can write

$$\mathbf{x}\ \alpha\ \mathbf{T(x, y)} = f^{-1}(\max(0, f(T(x,y)) - f(x))),$$
$$f(T(x,y)) = \min(f(0), f(x) + f(y)).$$

Let us now assume that $f(x)+f(y) > f(0)$, then $\mathbf{x}\ \alpha\ \mathbf{T(x, y)} = f^{-1}(f(0)-f(x))) > y$. If $f(x)+f(y) \le f(0)$, then $\mathbf{x}\ \alpha\ \mathbf{T(x, y)} = f^{-1}(f(y)) = y$.

Next, we recall and formulate necessary definitions and properties. The class of all the fuzzy subsets of the set E is denoted by L(E).

Definition 1. Let Q be a fuzzy relation from X to Y, $Q \in L(XxY)$, and R be a fuzzy relation from Y to Z, $R \in L(YxZ)$, then Sup-(t-norm) composition of R with Q,
$S = \mathbf{R} * \mathbf{Q}$, $S \in L(XxZ)$ is defined by

$$(\mathbf{R} * \mathbf{Q})(x,z) = \sup_{y \in Y} T(Q(x,y), R(y,z)) \quad \text{for all } (x,z) \in XxZ.$$

Definition 2. Let $Q \in L(XxY)$, $R \in L(YxZ)$, the α - composite fuzzy relations of Q and R, $S = \mathbf{Q}\ \alpha\ \mathbf{R}$, $S \in L(XxZ)$ is defined by

$$(\mathbf{Q}\ \alpha\ \mathbf{R})(x,z) = \inf_{y \in Y} (\mathbf{Q(x,y)}\ \alpha\ \mathbf{R(y,z)}) \quad \text{for all } (x,z) \in XxZ.$$

The general Problem of Sup-(t-norm) composition of fuzzy relation equations consists of finding the solution of $S = \mathbf{R} * \mathbf{Q}$, where S and Q (Problem I) or, S and R (Problem II) are known.

Theorem 3. If $Q \in L(XxY)$ and $R \in L(YxZ)$, then

$$R \subseteq \mathbf{Q}^{-1}\ \alpha\ (\mathbf{R} * \mathbf{Q})$$

Proof. Let $S = \mathbf{R} * \mathbf{Q} \in L(XxZ)$. From proposition 1, 2 of the Theorem 2, Definition 2 and Theorem 1, we can write

$$(\mathbf{Q}^{-1}\ \alpha\ \mathbf{S})(y,z) = \inf_{x \in X} (\mathbf{Q(x,y)}\ \alpha\ \mathbf{S(x,z)}).$$

$$S(x,z) = \sup_{t \in Y} T(\mathbf{Q}(x,t), \mathbf{R}(t,z)) = \max(T(\mathbf{Q}(x,y), \mathbf{R}(y,z))), \ \sup_{t \neq y} T(\mathbf{Q}(x,t), \mathbf{R}(t,z)) \ge$$

$$\ge T(\mathbf{Q}(x,y), \mathbf{R}(y,z)).$$

Consequently, $(\mathbf{Q}^{-1}\alpha\ \mathbf{S})(y,z) \ge \inf_{x \in X} (\mathbf{Q}(x,y)\ \alpha\ \mathbf{T}(\mathbf{Q}(x,y), \mathbf{R}(y,z))) \ge \mathbf{R}(y,z))$.

Theorem 4. If $Q \in L(XxY)$ and $S \in L(XxZ)$, then

$$(\mathbf{Q}^{-1}\ \alpha\ \mathbf{S}) * \mathbf{Q} \subseteq \mathbf{S}$$

Proof. From proposition 1,2 of the Theorem 2, Definition 2, Theorem 1, we have $((\mathbf{Q}^{-1}\alpha\ \mathbf{S}) * \mathbf{Q})(x,z) = \sup_{y\in Y} T(Q(x,y), (\mathbf{Q}\ \alpha\ \mathbf{S})(y,z))$,

$$(\mathbf{Q}\ \alpha\ \mathbf{S})(y,z)) = \inf_{t\in X} (\mathbf{Q}(t,y)\ \alpha\ \mathbf{S}(t,z)) =$$

$$= \min((\mathbf{Q}(x,y)\ \alpha\ \mathbf{S}(x,z)), \inf_{t\neq x}((\mathbf{Q}(t,y)\ \alpha\ \mathbf{S}(t,z)))) < \mathbf{Q}(x,y)\ \alpha\ \mathbf{S}(x,z).$$

Hence $((\mathbf{Q}^{-1}\ \alpha\ \mathbf{S}) * \mathbf{Q})(x,z) \leq \sup_{y\in Y} T(Q(x,y), (\mathbf{Q}\ \alpha\ \mathbf{S})(x,z)) \leq \mathbf{S}(x,z)$.

Our solution ot the Sup-(t-norm) fuzzy relation equations gives a greatest element of the sets of solution.

Indeed, if $S = \mathbf{R*Q}$, then denoting $R_{max} = \mathbf{Q}^{-1}\alpha\ \mathbf{S}$, from Theorems 3,4 we deduce: $R \subseteq R_{max}$, and $S = R * Q \subseteq R_{max} * Q \subseteq S$. So $S = R_{max} * Q$.

We solved Problem I, i.e. given $S \in (XxZ)$, $Q \in L(XxY)$, to know whether there exists $R \in L(YxZ)$, such that $S = \mathbf{R} * \mathbf{Q}$, it is sufficient to check that $S = R_{max} * Q$, where $R_{max} = \mathbf{Q}^{-1}\ \alpha\ \mathbf{S}$.

Note 3. If S, R are given, then to known whether there exists Q, such that $S = \mathbf{R} * \mathbf{Q}$, it is sufficient to check that $S = \mathbf{R} * \mathbf{Q}_{max}$, where $Q_{max} = (\mathbf{R}\ \alpha\ \mathbf{S}^{-1})^{-1}$ (it is Problem II and here we omitted the proof of this proposition).

Example 6. We will be use Q, R from the paper [2], but the purpose is to apply our results.

$$Q = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{cccc} Y_1 & Y_2 & Y_3 & Y_4 \\ 0.2 & 0 & 0.8 & 1 \\ 0.4 & 0.3 & 0 & 0.7 \\ 0.5 & 0.9 & 0.2 & 0 \end{array} \qquad R = \begin{array}{c} \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{array} \begin{array}{ccc} Z_1 & Z_2 & Z_3 \\ 0.3 & 0.5 & 0.2 \\ 0.8 & 1 & 0 \\ 0.7 & 0 & 0.5 \\ 0.6 & 0.3 & 1 \end{array}$$

For the *- composition we get $T(x,y) = xy$, $\mathbf{x}\ \alpha\ \mathbf{y} = \begin{cases} 1, \text{ if } x \leq y, \\ y/x, \text{ if } x>y. \end{cases}$

Then,

$$S = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{ccc} Z_1 & Z_2 & Z_3 \\ 0.6 & 0.3 & 1 \\ 0.56 & 0.3 & 0.7 \\ 0.73 & 0.9 & 0.1 \end{array} \qquad R_{max} = \begin{array}{c} \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{array} \begin{array}{ccc} Z_1 & Z_2 & Z_3 \\ 1 & 0.75 & 0.2 \\ 0.8 & 1 & 1/9 \\ 0.75 & 0.375 & 0.5 \\ 0.6 & 0.3 & 1 \end{array}$$

It is easy to verify that $\mathbf{R}_{max} * \mathbf{Q} = S$.

E.Sanchez [2] considered the following ε - operator

$$\mathbf{x}\ \varepsilon\ \mathbf{y} = \mathrm{Inf}(z \in [0;1] : \perp(x,z) \geq y) = \begin{cases} y, & \text{if } x < y, \\ 0, & \text{if } x \geq y. \end{cases} \quad \text{with } \perp(x,z) = \max(x,z),$$

and proposed a correspondingly method of resolution of the relational equations.

Here $\perp(x,y)$ is the t-conorm, which is associated with the t-norm by the De Morgan rule: $\perp(x,y) = n \circ T(n(x), n(y))$, with the negation $n(z) = 1\text{-}z$. In other words, $\perp$ and T are n-dual operators and its additive generators is associated as $g(z) = f \circ n(z)$.

Now, we formulate the ε - operator's definition with teminology of additive generator $g(z)$ of the t-conorm $\perp$.

Theorem 5. If $\perp(x,y) = g^{-1}(\min(g(1), g(x) + g(y)))$ then correspondingly ε-operator is

$$\mathbf{x}\ \varepsilon\ \mathbf{y} = \mathrm{Inf}(z \in [0;1] : \perp(x,z) \geq y) = g^{-1}(\max(0, g(y) - g(x))),$$

Proof. Similar to proof of Theorm 1.

Let $n(z)$ is a strict negation (involution) and not obligatory $n(z) = 1\text{-}z$.

Theorem 6. Let T and $\perp$ are dual in regard to involution $n(z)$

$$n \circ T(n(x), n(y)) = \perp(x,y),$$

then we have correspondingly two n-dual operators α and ε

$$n \circ (\mathbf{n}(x)\ \alpha\ \mathbf{n}(y)) = \mathbf{x}\ \varepsilon\ \mathbf{y}.$$

Proof. We remark, that for n-dual t-norm T and t-conorm $\perp$, $g = f \circ n$. Hence, $\mathbf{n}(x)\ \alpha\ \mathbf{n}(y) = f^{-1}(\max(0, f \circ n(y) - f \circ n(x))) = n \circ g^{-1}(\max(0, g(y) - g(x))) = n \circ (\mathbf{x}\ \varepsilon\ \mathbf{y})$.

Example 7. $T(x,y) = \max(0, x+y-1)$, $\perp(x,y) = \min(1, x+y)$ are dual in regard to involution $n(z)=1\text{-}z$. Then the following α and ε - operators are n- dual, $n(z) = 1\text{-}z$.

$$x\ \alpha\ y = \begin{cases} 1, & \text{if } x \leq y \\ y-x+1, & \text{if } x > y. \end{cases} \qquad x\ \varepsilon\ y = \begin{cases} 0, & \text{if } x \geq y \\ y-x, & \text{if } x < y \end{cases}$$

Note 4. Now, as in [2], we can to solve the Δ-composite fuzzy relation equation

$$S = \mathbf{R}\Delta\mathbf{Q},\ S \in L(XxZ),\ (\mathbf{R}\ \Delta\ \mathbf{Q})(x,z) = \inf_{y \in Y} \perp(Q(x,y), R(y,z)), \text{ for all } (x,z) \in XxZ.$$

It is based on the ε - operator, which is n -dual of the α - operator.

$$(\mathbf{Q}\ \varepsilon\ \mathbf{R})(x,z) = \sup_{y \in Y} (\mathbf{Q}(x,y)\ \varepsilon\ \mathbf{R}(y,z)), \text{ for all } (x,z) \in XxZ.$$

Thus, if $S = \mathbf{R}\Delta\mathbf{Q}$, and we known $S \in (XxZ)$, $Q \in L(XxY)$, then there exists $Rmin = \mathbf{Q}^{-1} \varepsilon\ \mathbf{S}$, and $Rmin\ \Delta\ Q = S$. If it is known $S \in L(XxZ)$, $R \in L(YxZ)$, then there exists
$Rmin = (\mathbf{R}\ \varepsilon\ \mathbf{S}^{-1})^{-1}$, and $Rmin\ \Delta\ Q = S$.

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