Lattice generated by fuzzy implications *

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We consider fuzzy implications generated by known examples of fuzzy implications. In recent papers we have long lists of formulas for fuzzy implication (cf. e.g. [3] or [10]). Our list is almost quite new.

Definition 1 ([1]). A function $I\colon [0,1]^2 \to [0,1]$ is called *fuzzy implication* if it is monotonic with respect to both variables and fulfils the binary implication truth table:

$$I(0,0) = I(0,1) = I(1,1) = 1, \quad I(1,0) = 0 \ . \tag{1}$$

Set of all fuzzy implications is denoted by FI.

Example 1. The most frequently used implication functions are usually listed with suitable author's name (cf. e.g. Dubois, Prade [6]).

1. Łukasiewicz implication [11]

$$I_{\mathrm{LK}}(x,y) = \min(1-x+y,1) = \begin{cases} 1, & \text{if } x \le y \\ 1-x+y, & \text{if } x > y \end{cases} ,$$

2. Gödel implication [9]

$$I_{\mathrm{GD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{if } x > y \end{cases} , \qquad I'_{\mathrm{GD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ 1-x, & \text{if } x > y \end{cases} ,$$

3. Reichenbach implication [12]

$$I_{\mathrm{RC}}(x,y) = 1 - x + xy \ ,$$

4. Dienes implication [5]

$$I_{\mathrm{DN}}(x,y) = \max(1-x,y) \ ,$$

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5. Goguen implication [8]

$$I_{\mathrm{GG}}(x,y)=\begin{cases}1, & \text{if } x\le y\\ \frac{y}{x}, & \text{if } x>y\end{cases}, \qquad I'_{\mathrm{GG}}(x,y)=\begin{cases}1, & \text{if } x\le y\\ \frac{1-x}{1-y}, & \text{if } x>y\end{cases},$$

6. Rescher implication [13]

$$I_{\mathrm{RS}}(x,y)=\begin{cases}1, & \text{if } x\le y\\ 0, & \text{if } x>y\end{cases},$$

for $x,y\in[0,1]$.

Our investigations were inspired by paper of Czogała, Łęski [4] were they ask for relative location of implications from Example 1.

Theorem 1 (cf. [1]). $(FI,\min,\max)$ *is a distributive lattice. In particular*

$$\forall_{I,J\in FI}\ (\min(I,J),\max(I,J)\in FI). \tag{2}$$

Theorem 2 (cf. [1]). *Lattice* $(FI,\min,\max)$ *has the least and the greatest element:*

$$I_0(x,y)=\begin{cases}1, & \textit{if } x=0 \textit{ or } y=1\\ 0, & \textit{if } x>0 \textit{ and } y<1\end{cases}, \qquad I_1(x,y)=\begin{cases}1, & \textit{if } x<1 \textit{ or } y>0\\ 0, & \textit{if } x=1 \textit{ and } y=0\end{cases},$$

for $x,y\in[0,1]$.

We look for the lattice generated by fuzzy implications from Example 1.

Theorem 3. *The distributive lattice generated by fuzzy implications from Example 1 consists of* 71 *elements with* 63 *generated implications* I_2–I_{64}*:*

$$I_2=I_{\mathrm{GG}}\vee I_{\mathrm{RC}},\quad I_2(x,y)=\begin{cases}1, & \textit{if } x\le y\\ \max(\frac{y}{x},1-x+xy), & \textit{if } x>y\end{cases},$$

$$I_3=I_{\mathrm{GG}}\wedge I_{\mathrm{RC}},\quad I_3(x,y)=\begin{cases}1-x+xy, & \textit{if } x\le y\\ \min(\frac{y}{x},1-x+xy), & \textit{if } x>y\end{cases},$$

$$I_4=I_{\mathrm{GD}}\vee I_{\mathrm{RC}},\quad I_4(x,y)=\begin{cases}1, & \textit{if } x\le y\\ 1-x+xy, & \textit{if } x>y\end{cases},$$

$$I_5=I_{\mathrm{GD}}\wedge I_{\mathrm{RC}},\quad I_5(x,y)=\begin{cases}1-x+xy, & \textit{if } x\le y\\ y, & \textit{if } x>y\end{cases},$$

$$I_6=I_{\mathrm{GG}}\vee I_{\mathrm{DN}},\quad I_6(x,y)=\begin{cases}1, & \textit{if } x\le y\\ \max(\frac{y}{x},1-x), & \textit{if } x>y\end{cases},$$

$$I_7=I_{\mathrm{GG}}\wedge I_{\mathrm{DN}},\quad I_7(x,y)=\begin{cases}\max(1-x,y), & \textit{if } x\le y\\ \min(\frac{y}{x},\max(1-x,y)), & \textit{if } x>y\end{cases},$$

$$I_8 = I_{\mathrm{GD}} \vee I_{\mathrm{DN}}, \quad I_8(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \max(1-x, y), & \textit{if } x > y \end{cases},$$

$$I_9 = I_{\mathrm{GD}} \wedge I_{\mathrm{DN}}, \quad I_9(x,y) = \begin{cases} \max(1-x, y), & \textit{if } x \le y \\ y, & \textit{if } x > y \end{cases},$$

$$I_{10} = I_6 \wedge I_{\mathrm{RC}}, \quad I_{10}(x,y) = \begin{cases} 1-x+xy, & \textit{if } x \le y \\ \max(1-x, \min(\frac{y}{x}, 1-x+xy)), & \textit{if } x > y \end{cases},$$

$$I_{11} = I_8 \wedge I_{\mathrm{RC}}, \quad I_{11}(x,y) = \begin{cases} 1-x+xy, & \textit{if } x \le y \\ \max(1-x, y), & \textit{if } x > y \end{cases},$$

$$I_{12} = I_4 \wedge I_{\mathrm{GG}}, \quad I_{12}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(\frac{y}{x}, 1-x+xy), & \textit{if } x > y \end{cases},$$

$$I_{13} = I_8 \wedge I_{\mathrm{GG}}, \quad I_{13}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(\frac{y}{x}, \max(1-x, y)), & \textit{if } x > y \end{cases},$$

$$I_{14} = I_4 \wedge I_6, \quad I_{14}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(1-x+xy, \max(\frac{y}{x}, 1-x)), & \textit{if } x > y \end{cases},$$

$$I_{15} = I_5 \vee I_7, \quad I_{15}(x,y) = \begin{cases} 1-x+xy, & \textit{if } x \le y \\ \min(\frac{y}{x}, \max(1-x, y)), & \textit{if } x > y \end{cases},$$

$$I_{16} = I_{\mathrm{RC}} \wedge I_{\mathrm{RS}}, \quad I_{16}(x,y) = \begin{cases} 1-x+xy, & \textit{if } x \le y \\ 0, & \textit{if } x > y \end{cases},$$

$$I_{17} = I_{\mathrm{DN}} \wedge I_{\mathrm{RS}}, \quad I_{17}(x,y) = \begin{cases} \max(1-x, y), & \textit{if } x \le y \\ 0, & \textit{if } x > y \end{cases},$$

$$I_{18} = I_{\mathrm{RC}} \vee I'_{\mathrm{GG}}, \quad I_{18}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \max(1-x+xy, \frac{1-x}{1-y}), & \textit{if } x > y \end{cases},$$

$$I_{19} = I_{\mathrm{RC}} \wedge I'_{\mathrm{GG}}, \quad I_{19}(x,y) = \begin{cases} 1-x+xy, & \textit{if } x \le y \\ \min(1-x+xy, \frac{1-x}{1-y}), & \textit{if } x > y \end{cases},$$

$$I_{20} = I_{\mathrm{DN}} \vee I'_{\mathrm{GG}}, \quad I_{20}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \max(y, \frac{1-x}{1-y}), & \textit{if } x > y \end{cases},$$

$$I_{21} = I_{\mathrm{DN}} \wedge I'_{\mathrm{GG}}, \quad I_{21}(x,y) = \begin{cases} \max(1-x, y), & \textit{if } x \le y \\ \min(\frac{1-x}{1-y}, \max(1-x, y)), & \textit{if } x > y \end{cases},$$

$$I_{22} = I_{19} \vee I_{\mathrm{DN}}, \quad I_{22}(x,y) = \begin{cases} 1-x+xy, & \textit{if } x \le y \\ \max(y, \min(1-x+xy, \frac{1-x}{1-y})), & \textit{if } x > y \end{cases},$$

$$I_{23} = I_{19} \vee I'_{\mathrm{GD}}, \quad I_{23}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(1-x+xy, \frac{1-x}{1-y}), & \textit{if } x > y \end{cases},$$

$$I_{24} = I_{21} \vee I'_{\mathrm{GD}}, \quad I_{24}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(\frac{1-x}{1-y}, \max(1-x, y)), & \textit{if } x > y \end{cases},$$

$$I_{25} = I_{22} \vee I_{23}, \quad I_{25}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(1-x+xy, \max(y, \frac{1-x}{1-y})), & \textit{if } x > y \end{cases},$$

$$I_{26} = I_{\mathrm{RC}} \wedge I'_{\mathrm{GD}}, \quad I_{26}(x,y) = \begin{cases} 1-x+xy, & \textit{if } x \le y \\ 1-x, & \textit{if } x > y \end{cases},$$

$$I_{27} = I_{21} \vee I_{26}, \quad I_{27}(x,y) = \begin{cases} 1-x+xy, & \textit{if } x \le y \\ \min(\frac{1-x}{1-y}, \max(1-x, y)), & \textit{if } x > y \end{cases},$$

$$I_{28} = I_{\mathrm{DN}} \wedge I'_{\mathrm{GD}}, \quad I_{28}(x,y) = \begin{cases} \max(1-x, y), & \textit{if } x \le y \\ 1-x, & \textit{if } x > y \end{cases},$$

$$I_{29} = I_{\mathrm{GG}} \vee I'_{\mathrm{GG}}, \quad I_{29}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \max(\frac{y}{x}, \frac{1-x}{1-y}), & \textit{if } x > y \end{cases},$$

$$I_{30} = I_{\mathrm{GG}} \wedge I'_{\mathrm{GG}}, \quad I_{30}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(\frac{y}{x}, \frac{1-x}{1-y}), & \textit{if } x > y \end{cases},$$

$$I_{31} = I_{\mathrm{GD}} \wedge I'_{\mathrm{GD}}, \quad I_{31}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(1-x, y), & \textit{if } x > y \end{cases}.$$

$$I_{32} = I_{\mathrm{RC}} \vee I_{29}, \quad I_{32}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \max(1-x+xy, \frac{y}{x}, \frac{1-x}{1-y}), & \textit{if } x > y \end{cases},$$

$$I_{33} = I_{\mathrm{RC}} \wedge I_{29}, \quad I_{33}(x,y) = \begin{cases} 1-x+xy, & \textit{if } x \le y \\ \min(1-x+xy, \max(\frac{y}{x}, \frac{1-x}{1-y})), & \textit{if } x > y \end{cases},$$

$$I_{34} = I_{4} \wedge I_{29}, \quad I_{34}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(1-x+xy, \max(\frac{y}{x}, \frac{1-x}{1-y})), & \textit{if } x > y \end{cases},$$

$$I_{35} = I_{8} \vee I_{30}, \quad I_{35}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \max(1-x, y, \min(\frac{y}{x}, \frac{1-x}{1-y})), & \textit{if } x > y \end{cases},$$

$$I_{36} = I_{8} \wedge I_{30}, \quad I_{36}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(\max(1-x, y), \frac{y}{x}, \frac{1-x}{1-y}), & \textit{if } x > y \end{cases},$$

$$I_{37} = I_{4} \wedge I_{35}, \quad I_{37}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \max(1-x, y, \min(1-x+xy, \frac{y}{x}, \frac{1-x}{1-y})), & \textit{if } x > y \end{cases},$$

$$I_{38} = I_{\mathrm{RC}} \wedge I_{35}, \quad I_{38}(x,y) = \begin{cases} 1-x+xy, & \textit{if } x \le y \\ \max(1-x, y, \min(1-x+xy, \frac{y}{x}, \frac{1-x}{1-y})), & \textit{if } x > y \end{cases},$$

$$I_{39} = I_{4} \wedge I_{30}, \quad I_{39}(x,y) = \begin{cases} 1, & \textit{if } x \le y \\ \min(1-x+xy, \frac{y}{x}, \frac{1-x}{1-y}), & \textit{if } x > y \end{cases},$$

$$I_{40} = I_{\mathrm{RC}} \wedge I_{30}, \quad I_{40}(x,y) = \begin{cases} 1 - x + xy, & \text{if } x \le y \\ \min(1 - x + xy, \frac{y}{x}, \frac{1-x}{1-y}), & \text{if } x > y \end{cases},$$

$$I_{41} = I_{\mathrm{RC}} \wedge I_{36}, \quad I_{41}(x,y) = \begin{cases} 1 - x + xy, & \text{if } x \le y \\ \min(\max(1 - x, y), \frac{y}{x}, \frac{1-x}{1-y}), & \text{if } x > y \end{cases},$$

$$I_{42} = I_{\mathrm{DN}} \wedge I_{36}, \quad I_{42}(x,y) = \begin{cases} \max(1 - x, y), & \text{if } x \le y \\ \min(\max(1 - x, y), \frac{y}{x}, \frac{1-x}{1-y}), & \text{if } x > y \end{cases},$$

$$I_{43} = I_{\mathrm{RC}} \wedge I_{31}, \quad I_{43}(x,y) = \begin{cases} 1 - x + xy, & \text{if } x \le y \\ \min(1 - x, y), & \text{if } x > y \end{cases},$$

$$I_{44} = I_{\mathrm{DN}} \wedge I_{31}, \quad I_{44}(x,y) = \begin{cases} \max(1 - x, y), & \text{if } x \le y \\ \min(1 - x, y), & \text{if } x > y \end{cases},$$

$$I_{45} = I_{\mathrm{RC}} \vee I_{30}, \quad I_{45}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \max(1 - x + xy, \min(\frac{y}{x}, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

$$I_{46} = I_{29} \wedge I_{45}, \quad I_{46}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \min(\max(1 - x + xy, \min(\frac{y}{x}, \frac{1-x}{1-y})), \max(\frac{y}{x}, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

$$I_{47} = I_{\mathrm{GG}} \wedge I_{18}, \quad I_{47}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \min(\frac{y}{x}, \max(1 - x + xy, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

$$I_{48} = I'_{\mathrm{GG}} \wedge I_{2}, \quad I_{48}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \min(\frac{1-x}{1-y}, \max(1 - x + xy, \frac{y}{x})), & \text{if } x > y \end{cases},$$

$$I_{49} = I_{\mathrm{GG}} \wedge I'_{\mathrm{GD}}, \quad I_{49}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \min(\frac{y}{x}, 1 - x), & \text{if } x > y \end{cases},$$

$$I_{50} = I'_{\mathrm{GG}} \wedge I_{\mathrm{GD}}, \quad I_{50}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \min(y, \frac{1-x}{1-y}), & \text{if } x > y \end{cases},$$

$$I_{51} = I_{\mathrm{DN}} \wedge I_{49}, \quad I_{51}(x,y) = \begin{cases} \max(1 - x, y), & \text{if } x \leqslant y \\ \min(\frac{y}{x}, 1 - x), & \text{if } x > y \end{cases},$$

$$I_{52} = I_{\mathrm{DN}} \wedge I_{50}, \quad I_{52}(x,y) = \begin{cases} \max(1 - x, y), & \text{if } x \leqslant y \\ \min(y, \frac{1-x}{1-y}), & \text{if } x > y \end{cases},$$

$$I_{53} = I_{\mathrm{GG}} \vee I_{19}, \quad I_{53}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \max(\frac{y}{x}, \min(1 - x + xy, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

$$I_{54} = I'_{\mathrm{GG}} \vee I_{3}, \quad I_{54}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \max(\frac{1-x}{1-y}, \min(1 - x + xy, \frac{y}{x})), & \text{if } x > y \end{cases},$$

$$I_{55} = I_{\mathrm{GG}} \wedge I_{26}, \quad I_{55}(x,y) = \begin{cases} 1 - x + xy, & \text{if } x \leqslant y \\ \min(\frac{y}{x}, 1 - x), & \text{if } x > y \end{cases},$$

$$I_{56} = I'_{\mathrm{GG}} \wedge I_5, \quad I_{56}(x,y) = \begin{cases} 1 - x + xy, & \text{if } x \leqslant y \\ \min(y, \frac{1-x}{1-y}), & \text{if } x > y \end{cases},$$

$$I_{57} = I_{\mathrm{GG}} \wedge I_{20}, \quad I_{57}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \max(y, \min(\frac{y}{x}, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

$$I_{58} = I'_{\mathrm{GG}} \wedge I_6, \quad I_{58}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \max(1-x, \min(\frac{y}{x}, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

$$I_{59} = I_2 \wedge I_{20}, \quad I_{59}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \min(\max(1-x+xy, \frac{y}{x}), \max(y, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

$$I_{60} = I_6 \wedge I_{18}, \quad I_{60}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \min(\max(1-x, \frac{y}{x}), \max(1-x+xy, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

$$I_{61} = I_3 \wedge I_{20}, \quad I_{61}(x,y) = \begin{cases} 1-x+xy, & \text{if } x \leqslant y \\ \min(\min(1-x+xy, \frac{y}{x}), \max(y, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

$$I_{62} = I_6 \wedge I_{19}, \quad I_{62}(x,y) = \begin{cases} 1-x+xy, & \text{if } x \leqslant y \\ \min(\min(1-x+xy, \frac{1-x}{1-y}), \max(1-x, \frac{y}{x})), & \text{if } x > y \end{cases},$$

$$I_{63} = I_4 \wedge I_{57}, \quad I_{63}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \max(y, \min(1-x+xy, \frac{y}{x}, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

$$I_{64} = I_4 \wedge I_{58}, \quad I_{64}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y \\ \max(1-x, \min(1-x+xy, \frac{y}{x}, \frac{1-x}{1-y})), & \text{if } x > y \end{cases},$$

for $x, y \in [0,1]$.

The rich family of the above presented fuzzy implications will be examined later in connection with fuzzy logic and approximate reasoning. We obtain the most complete list of monotonic fuzzy implications. However, it must be pointed out that a few members of this family were defined and examined in recent literature. E.g. I'_{GG} and I'_{GD} are mentioned in [6], p.157, I_8 appears in [7] p.31 and $I_{30} = R_{nd}$, $I_{32} = R_{nb}$ are examined in [14].

Theorem 4. *The lattice generated by fuzzy implications from Example 1 has the following Hasse diagram [2] (using numbering from Theorem 3):*

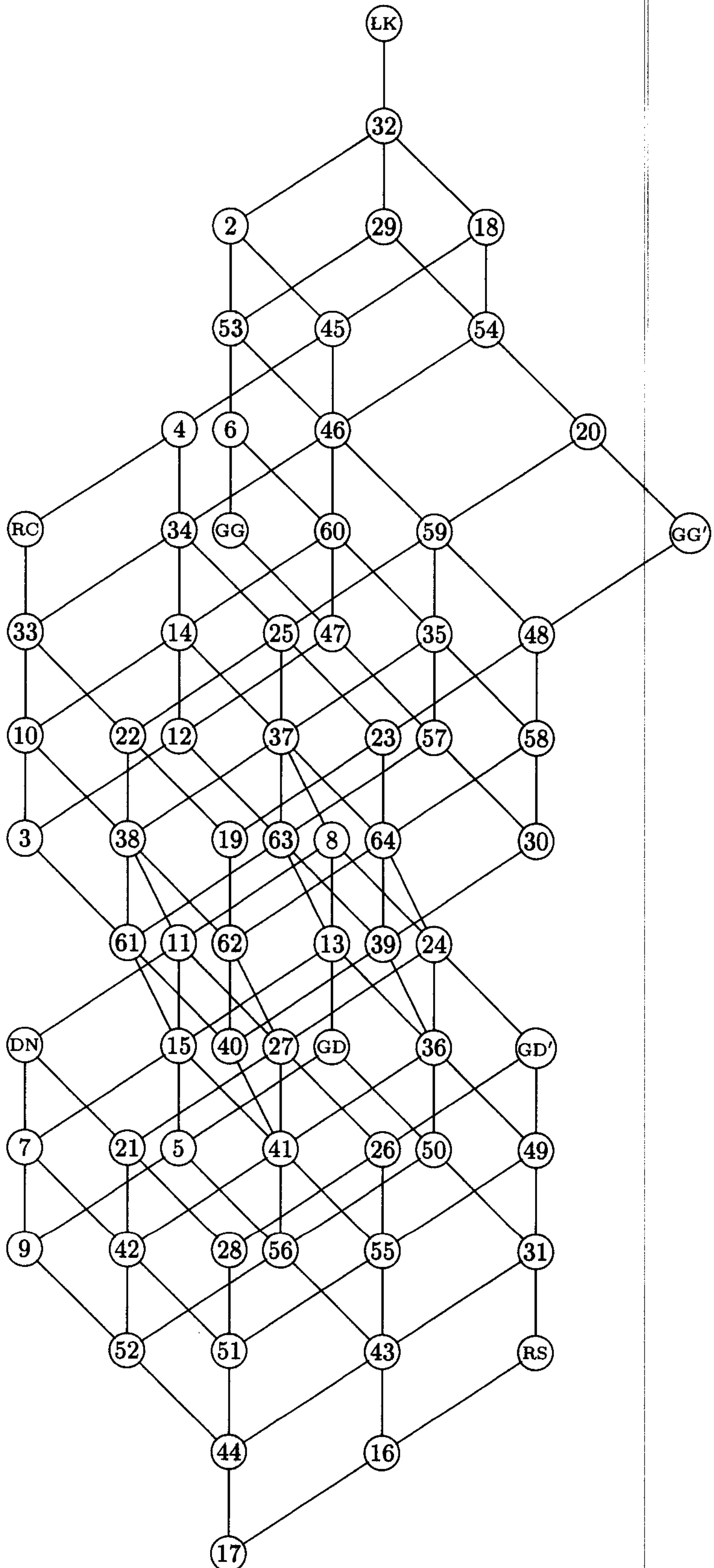

Fig. 1. Lattice generated by implications from Example 1

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