Fuzzy Rings and Fuzzy Ideals

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Abstract: Basing on the theory of fuzzy space and fuzzy binary operation, we introduce the concept of fuzzy ring and fuzzy ideal. This gives a new approach to study fuzzy ring theory.

Keywords: Fuzzy space; Fuzzy binary operation; Fuzzy ring; Fuzzy ideal.

1. Preliminary

Now we recall some definitions and results, which will be used in the sequel. For detail we refer to [1,2].

Throughout the paper, unless otherwise stated, I always represents the closed unit interval [0,1] of real numbers; L, K denote arbitrary lattices. L□K indicates the lattice L×K with the partial order defined by

(i) $(r_1, r_2) \le (s_1, s_2)$, *iff* $r_1 \le s_1$ *and* $r_2 \le s_2$, *where* $s_1 \ne 0$ *and* $s_2 \ne 0$,

(ii) $(0,0) = (s_1, s_2)$, *whenever* $s_1 = 0$ *or* $s_2 = 0$.

Definition1.1 Let X be an ordinary set and L be a completely distributive lattice with maximal and minimal elements denoted by 1,0, respectively. The fuzzy space, denoted by (X,L), is defined as follows:

$$(X, L) = \{(x, L); x \in X\},$$

Where (x,L) is called a fuzzy element and it is given by the relation

$$(x, L) = \{(x, r); r \in L\}.$$

The sublattice $l \subset$L is called an M-sublattice of L, if it has at least one element more than 0 and a maximal element denoted by 1_l.

Definition1.2 The subspace U of the fuzzy space (X,L) is defined as follows:

$$U = \{(x, u_x); x \in U_0\},$$

Where U_0 is an ordinary subset of X and $u_x (x \in U_0)$ is an M-sublattice of L.

$u_x (x \in U_0)$ is called the set of membership values of x in the fuzzy subspace U.

The subset U_0 is called the support of U in X and is denoted by S(U).

The empty fuzzy subspace $\{(x, \varnothing_x); x \in \varnothing\}$ will be denoted, also, by $\varnothing$. This means that $\varnothing_x = \{0\}$ for all $x \in X$.

Proposition1.1 Let A be a fuzzy subset of X and

$$H_0(A) = \{(x, \{0, A(x)\}); A(x) \ne 0\},$$

$$\underline{H}(A) = \{(x, [0, A(x)]); A(x) \ne 0\},$$

$$\overline{H}(A) = \{(x, \{0\} \cup [A(x), 1]); A(x) \ne 0\}.$$

Then $H_0(A), \underline{H}(A), \overline{H}(A)$ are all the fuzzy subspace of fuzzy space (X,I). We call them fuzzy subspaces induced by A.

Definition1.3 Let (X,L) and (Y,K) be two fuzzy spaces. We call the fuzzy space (X×Y,L□K) the fuzzy Cartesian product of the fuzzy spaces (X,L) and (Y,K), denoted by (X,L)□(Y,K). If I=L=K, then for simplicity we write (X,L)□(Y,K)=X□Y.

Definition1.4 The fuzzy Cartesian product of the L-fuzzy subsets A of X and the K-fuzzy subset B of Y is an L□K-fuzzy subset, denoted by A□B, is defined as follows:

$$A\square B=\{((x,y),(A(x),B(y)));x\in X \text{ and } y\in Y\}.$$

It's clear that A□B∈(X,L)□(Y,K).

Definition1.5 The fuzzy function $\underline{F}$ from the fuzzy space (X,L) into the fuzzy space (Y,K) is defined as an ordered pair $\underline{F}=(F,\{f_x\}_{x\in X})$, where F is a function from X into Y, and $\{f_x\}_{x\in X}$ is a family of onto functions $f_x:L\to K$, satisfying the conditions:

(i) f_x is nondecreasing on L,

(ii) $f_x(0)=0$ and $f_x(1)=1$.

We write $\underline{F}=(F,\{f_x\}_{x\in X})$:(X,L)→(Y,K) and we call the functions $f_x, x\in X$, the comembership functions associated to $\underline{F}$. A fuzzy function $\underline{F}=(F,f_x)$ is said to be uniform if the comembership functions f_x are identical for all $x\in X$.

Proposition1.2 Let $\underline{F}=(F,f_x)$, which has continuous comembership $f_x, x\in X$, be a function from (X,I) to (Y,I), and $U=\{(x,u_x);x\in U_0\}$ is a fuzzy subspace of (X,I), then

$$\underline{F}(U)=\{\underline{F}(x,u_x);(x,u_x)\in U\}=\{(F(x),f_x(u_x));x\in S(U)\}$$

is a fuzzy subspace of (Y,I) iff $f_x(u_x)=f_{x'}(u_{x'})$, for any F(x)=F(x').

Proposition1.3 Let $\underline{F}$ be a function from (X,I) to (Y,I), then $\underline{F}$ defines a function from the fuzzy subspace $U=\{(x,u_x);x\in U_0\}$ of (X,I) to the fuzzy subspace $V=\{(x,v_x);x\in V_0\}$ of (Y,I) iff $F(U_0)\subset V_0$ and $f_x(u_x)=v_{F(x)}$. Specially, if $U=H_0(A)$ and $V=H_0(B)$, for some fuzzy subsets A of X and B of Y, then $\underline{F}(H_0(A))=H_0(B)$ iff $\underline{F}(A_0)\subset B_0$ and $f_x(A(x))=B(F(x))$, $x\in A_0$, where A_0 and B_0 denote the support sets of A and B, respectively.

2. Fuzzy Ring and Fuzzy Subring

Definition2.1 A fuzzy binary operation $\underline{F}=(F,f_{xy})$ on the fuzzy space (X,I) is a fuzzy function from (X,I)□(X,I) to (X,I), i.e.

$\underline{F}=(F,f_{xy})$: (X×X,I□I)→(X,I), where F:X×X→X with onto comembership functions f_{xy}: I□I→I, which satisfy $f_{xy}(r,s)\neq 0$ if r≠0 and s≠0.

In the following we shall use the notions:

$$F(x,y)=xFy \text{ and } f_{xy}(r,s)=rf_{xy}s.$$

Thus, for any two fuzzy elements (x,I), (y,I) of (X,I), we have

(x,I) $\underline{F}$ (y,I)=: $\underline{F}$ ((x,I)□(y,I))= $\underline{F}$ ((x,y),I□I)=(F(x,y), f_{xy} (I□I))=:(xFy,I).

It is clear that F is a binary operation on X.

The fuzzy binary operation $\underline{F}=(F,f_{xy})$ on (X,I) is said to be uniform if the associated comembership functions f_{xy} are identical for all x,y∈X.

Definition2.2 A fuzzy groupoid is a fuzzy space (X,I) with a binary operation $\underline{F}=(F,f_{xy})$.

A fuzzy semigroup is a fuzzy groupoid that is associate.

A fuzzy monoid is a fuzzy semigroup which admits an identity element (e,I) such that for every (x,I)∈(X,I) we have

$$(x,I)\underline{F}(e,I)=(e,I)\underline{F}(x,I)=(x,I).$$

A fuzzy group is a fuzzy monoid in which every fuzzy element (x,I) has an inverse $(x,I)^{-1}$ such that

$$(x,I)\underline{F}(x,I)^{-1}=(x,I)^{-1}\underline{F}(x,I)=(e,I).$$

A fuzzy Abelian group is a fuzzy group if $\underline{F}$ is communicative.

Definition2.3 Let $\underline{F}^{+}=(F^{+},f_{xy}^{+})$ and $\underline{F}^{\bullet}=(F^{\bullet},f_{xy}^{\bullet})$ are two fuzzy binary operations on the fuzzy space (X,I). We call $((X,I),\underline{F}^{+},\underline{F}^{\bullet})$ a fuzzy ring if following conditions holds:

(i) $((X,I),\underline{F}^{+})$ is a fuzzy Abelian group,

(ii) $((X,I),\underline{F}^{\bullet})$ is a fuzzy semigroup,

(iii) The distributive laws

$$(x,I)\underline{F}^{\bullet}((y,I)\underline{F}^{+}(z,I))=((x,I)\underline{F}^{\bullet}(y,I))\underline{F}^{+}((x,I)\underline{F}^{\bullet}(z,I)),$$

$$((y,I)\underline{F}^{+}(z,I))\underline{F}^{\bullet}(x,I)=((y,I)\underline{F}^{\bullet}(x,I))\underline{F}^{+}((z,I)\underline{F}^{\bullet}(x,I))$$

holds for all (x,I), (y,I), (z,I)∈(X,I). The fuzzy ring $((X,I),\underline{F}^{+},\underline{F}^{\bullet})$ is uniform if $\underline{F}^{+}$ and $\underline{F}^{\bullet}$ are all uniform.

Theorem2.1 Let $((X,I),\underline{F}^{+},\underline{F}^{\bullet})$ be a fuzzy ring, then $(X,F^{+},F^{\bullet})$ is an ordinary ring, and $((X,I),\underline{F}^{+},\underline{F}^{\bullet})$ is isomorphic to $(X,F^{+},F^{\bullet})$ under the correspondence $\varphi:x\to(x.I)$.

The proof is straightforward.

Definition2.4 The fuzzy subspace $U=\{(x,u_x);x\in U_0\}$ is said to be a fuzzy subring of the fuzzy ring $((X,I),\underline{F}^{+},\underline{F}^{\bullet})$, if

(i) $\underline{F}^{+}$ and $\underline{F}^{\bullet}$ are closed on the fuzzy subspace U,

(ii) $(U,\underline{F}^{+},\underline{F}^{\bullet})$ satisfies the axioms of the ordinary ring.

Theorem2.2 The fuzzy subspace $U=\{(x,u_x);x\in U_0\}$ is a fuzzy subring of the fuzzy ring $((X,I),\underline{F}^{+},\underline{F}^{\bullet})$ iff

(i) $(U_0,F^{+},F^{\bullet})$ is an ordinary subring of $(X,F^{+},F^{\bullet})$,

(ii) $f_{xy}^{+}(u_x,u_y)=u_{xF^{+}y}$, $f_{xy}^{\bullet}(u_x,u_y)=u_{xF^{\bullet}y}$, $x,y\in U_0$.

Proof If U is a fuzzy subring, then (i) is immediate from Definition2.4 .

(ii) Because

$$(x,u_x)\underline{F}^+(y,u_y)=(xF^+y,f_{xy}^+(u_x,u_y))\in U,$$
$$(x,u_x)\underline{F}^\bullet(y,u_y)=(xF^\bullet y,f_{xy}^\bullet(u_x,u_y))\in U,$$

so $f_{xy}^+(u_x,u_y)=u_{xF^+y}$, $f_{xy}^\bullet(u_x,u_y)=u_{xF^\bullet y}$.

Conversely, if (i) and (ii) holds, then for any (x,u_x), $(y,u_y)\in U$,

$$(x,u_x)\underline{F}^+(y,u_y)=(xF^+y,f_{xy}^+(u_x,u_y)=(xF^+y,u_{xF^+y})\in U.$$

Similarly, we have $(x,u_x)\underline{F}^\bullet(y,u_y)=(xF^\bullet y,f_{xy}^\bullet(u_x,u_y)=(xF^\bullet y,u_{xF^\bullet y})\in U$, by $(U_0,F^+,F^\bullet)$ is an ordinary subring of $(X,F^+,F^\bullet)$, it is easy to verify that $(U,\underline{F}^+,\underline{F}^\bullet)$ is a fuzzy subring of $((X,I),\underline{F}^+,\underline{F}^\bullet)$. These complete the proof. □

Definition2.5 If $H_0(A)$, $\underline{H}(A)$ and $\overline{H}(A)$ are fuzzy subring of $((X,I),\underline{F}^+,\underline{F}^\bullet)$, then we say the fuzzy subset A of X induces fuzzy subring of $((X,I),\underline{F}^+,\underline{F}^\bullet)$.

Theorem2.3 Let the fuzzy subspace U is induced by a fuzzy subset A of X, then $(U,\underline{F}^+,\underline{F}^\bullet)$ is a fuzzy subring iff (i) $(A_0,F^+,F^\bullet)$ is an ordinary ring, (ii) $f_{xy}^+(A(x),A(y))=A(xF^+y)$, $f_{xy}^\bullet(A(x),A(y))=A(xF^\bullet y)$, for all $A(x)\neq 0$ and $A(y)\neq 0$.

Proof. We prove the result for $U=H_0(A)$ only.

If U is a fuzzy subring, by the Theorem2.2 we know (i) holds. And

$$f_{xy}^+(u_x,u_y)=f_{xy}^+(\{0,A(x)\},\{0,A(y)\})=u_{xF^+y}=\{0,A(xF^+y)\},$$

so $f_{xy}^+(A(x),A(y))=A(xF^+y)$. Similarly, we have $f_{xy}^\bullet(A(x),A(y))=A(xF^\bullet y)$.

Conversely, if (i) and (ii) hold, then

$$f_{xy}^+(u_x,u_y)=f_{xy}^+(\{0,A(x)\},\{0,A(y)\})=\{(0,0),f_{xy}^+(A(x),A(y))\}=\{0,A(xF^+y)\}=u_{xF^+y}$$
$$f_{xy}^\bullet(u_x,u_y)=f_{xy}^\bullet(\{0,A(x)\},\{0,A(y)\})=\{(0,0),f_{xy}^\bullet(A(x),A(y))\}=\{0,A(xF^\bullet y)\}=u_{xF^\bullet y}$$

by Theorem2.2, $(U,\underline{F}^+,\underline{F}^\bullet)$ is a fuzzy subring. □

Theorem2.4 (i) Let $((X,I),\underline{F}^+,\underline{F}^\bullet)$ be a uniform fuzzy ring and let the comembership function f^+ and $f^\bullet$ have the t-norm property and $f^+=f^\bullet=f$. Then every subset A of X, which induces fuzzy subring, is a classical fuzzy subring of the ring $(X,F^+,F^\bullet)$.

(ii) If $(Y,F^+,F^\bullet)$ is an ordinary subring of the ring $(X,F^+,F^\bullet)$, then every fuzzy subset A, for which $A_0=\{x\in X,A(x)\neq 0\}=Y$, induces fuzzy subrings of ring $((X,I),\underline{G}^+,\underline{G}^\bullet)$, such that $\underline{G}^+=(G^+,g_{xy}^+)$, $\underline{G}^\bullet=(G^\bullet,g_{xy}^\bullet)$, where $G^+=F^+$, $G^\bullet=F^\bullet$, and $g_{xy}^+(r,s)$, $g_{xy}^\bullet(r,s)$ are suitable comembership functions.

Proof (i) If A induces fuzzy subrings of the fuzzy ring $((X,I),\underline{F}^+,\underline{F}^\bullet)$, then we have

$f_{xy}^{+}(A(x),A(y))=A(xF^{+}y)$, $f_{xy}^{\bullet}(A(x),A(y))=A(xF^{\bullet}y)$,
for all $A(x)\neq 0$ and $A(x)\neq 0$.

For f_{xy}^{+}, $f_{xy}^{\bullet}$ are uniform and $f^{+}=f^{\bullet}=f$, above equalities can be writed as following:

$f(A(x),A(y))=A(xF^{+}y)$, $f(A(x),A(y))=A(xF^{\bullet}y)$, for $A(x)\neq 0$ and $A(x)\neq 0$.

This means that if the fuzzy subset A induces fuzzy subrings, then it satisfying the inequalities:

$f(A(x),A(y))\leq A(xF^{+}y)$, $f(A(x),A(y))\leq A(xF^{\bullet}y)$, for any x,y in X.

Therefore, A is a classical fuzzy subring.

(ii) Let $(Y,F^{+},F^{\bullet})$ be a crisp subring of $(X,F^{+},F^{\bullet})$, A be a fuzzy subset for which $A_0=\{x\in X, A(x)\neq 0\}=Y$, and f be a given t-norm.

Now we define the fuzzy ring $((X,I),\underline{G}^{+},\underline{G}^{\bullet})$ as

$$\underline{G}^{+}=(G^{+},g_{xy}^{+}),\quad \underline{G}^{\bullet}=(G^{\bullet},g_{xy}^{\bullet}),$$

where $G^{+}=F^{+},G^{\bullet}=F^{\bullet}$ and $g_{xy}^{+}(r,s)=h_{xy}^{+}(f(r,s))$, $g_{xy}^{\bullet}(r,s)=h_{xy}^{\bullet}(f(r,s))$, where

$$h_{xy}^{+}(k)=\begin{cases}\dfrac{A(xF^{+}y)}{f(A(x),A(y))}k & \text{if } k\leq f(A(x),A(y))\\[2ex] 1+\dfrac{1-A(xF^{+}y)}{1-f(A(x),A(y)}(k-1) & \text{if } k> f(A(x),A(y))\end{cases},$$

$$h_{xy}^{\bullet}(k)=\begin{cases}\dfrac{A(xF^{\bullet}y)}{f(A(x),A(y))}k & \text{if } k\leq f(A(x),A(y))\\[2ex] 1+\dfrac{1-A(xF^{\bullet}y)}{1-f(A(x),A(y)}(k-1) & \text{if } k> f(A(x),A(y))\end{cases}.$$

It is obvious that $g_{xy}^{+}(r,s), g_{xy}^{\bullet}(r,s)$ are continuous comembership functions and $g_{xy}^{+}(r,s)=0$ ($g_{xy}^{\bullet}(r,s)=0$) iff r=0 or s=0. Therefore, $\underline{G}^{+},\underline{G}^{\bullet}$ are fuzzy binary operations on X.

It is not difficulty to verify that $((X,I),\underline{G}^{+},\underline{G}^{\bullet})$ is a fuzzy ring. Using the property of the t-norm function *f(r,s)*, we get: If $A(x)\neq 0$ and $A(y)\neq 0$, then $f(A(x),A(y))\neq 0$, and

$g_{xy}^{+}(A(x),A(y))=h_{xy}^{+}(f(A(x),A(y)))=A(xF^{+}y)$,

$g_{xy}^{\bullet}(A(x),A(y))=h_{xy}^{\bullet}(f(A(x),A(y)))=A(xF^{\bullet}y)$.

Therefore, A induces fuzzy subrings of the fuzzy ring $((X,I),\underline{G}^{+},\underline{G}^{\bullet})$. □

Corollary2.5 Every classical fuzzy subring A of ring $(X,F^{+},F^{\bullet})$ induces fuzzy subrings relative to some fuzzy ring $((X,I),\underline{G}^{+},\underline{G}^{\bullet})$.

3. Fuzzy Ideal

Definition3.1 Let $U=\{(a,u_a);a\in U_0\}$ be a fuzzy subring of fuzzy ring $((X,I),\underline{F}^+,\underline{F}^\bullet)$. We call U the fuzzy ideal of $((X,I),\underline{F}^+,\underline{F}^\bullet)$, if it satisfy the following conditions:

(i) U is a fuzzy normal subgroup of the fuzzy group $((X,I),\underline{F}^+)$ (see [2]);

(ii) For any $(x,I)\in(X,I)$ and $(a,u_a)\in U$,

$$(x,I)\underline{F}^\bullet(a,u_a)\in U,\ (a,u_a)\underline{F}^\bullet(x,I)\in U.$$

Propsition3.1 The condition (ii) of definition3.1 can be replaced with the following condition

(ii)' For any $x\in X$ and $a\in U_0$,

$$f^\bullet_{xF^\bullet a}(I,u_a)=u_{xF^\bullet a},\ f^\bullet_{aF^\bullet x}(u_a,I)=u_{aF^\bullet x}.$$

Let U be a fuzzy ideal of fuzzy ring $((X,I),\underline{F}^+,\underline{F}^\bullet)$, from [2] we know the set of all cosets of U to $\underline{F}^+$, defined by $((X,I),\underline{F}^+)/U=\{(x,I)\underline{F}^+U;(x,I)\in(X,I)\}$, forms a group under the fuzzy binary operation $\underline{F}^+$:

$$((x,I)\underline{F}^+U)\underline{F}^+((y,I)\underline{F}^+U)=(xF^+y,I)\underline{F}^+U.$$

For any $x,y\in X$, if U is a fuzzy ideal, then

$$((x,I)\underline{F}^+(a,u_a))\underline{F}^\bullet((y,I)\underline{F}^+(b,u_b))=(xF^\bullet y,I)\underline{F}^+((x,I)\underline{F}^\bullet(b,u_b))\underline{F}^+((a,u_a)\underline{F}^\bullet(y,I))\underline{F}^+((a,u_a)\underline{F}^\bullet(b,u_b))$$

$\in(xF^\bullet y,I)\underline{F}^+U$, this follows that

$$((x,I)\underline{F}^+U)\underline{F}^\bullet((y,I)\underline{F}^+U)=(xF^\bullet y,I)\underline{F}^+U$$

is a binary operation of $((X,I),\underline{F}^+)/U$, denoted by $\underline{F}^\bullet$ still. Therefore, it follows

Theorem3.2 If $U=\{(a,u_a);a\in U_0\}$ is a fuzzy ideal of the fuzzy ring $((X,I),\underline{F}^+,\underline{F}^\bullet)$, then the set of all cosets of U to $\underline{F}^+$ forms a ring under the fuzzy binary operations $\underline{F}^+,\underline{F}^\bullet$, denoted by $((X,I),\underline{F}^+,\underline{F}^\bullet)/U$, we call it the fuzzy factor ring.

Theorem3.3 The fuzzy factor ring $((X,I),\underline{F}^+,\underline{F}^\bullet)/U$ is isomorphic to the quotient ring $(X,F^+,F^\bullet)/U_0$ by the correspondence

$$(x,I)\underline{F}^+U\leftrightarrow xF^+U_0.$$

Referees

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