The lattice structure of fuzzy K-algebra

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Abstract

In this paper, we study the lattice structure of fuzzy K-subalgebras with the technique of nested set and obtain that the lattice of all fuzzy K-subalgebras is modular.

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1. Preliminaries

We recall some definitions and results first. K always represents a communicative ring with unit element 1; A denotes a K-algebra; sub(A) denotes the set of all K-subalgebras of A; I(A) denotes the set of all algebra ideals of A.

Proposition1.1 *sub(A) forms a complete modular lattice with maximal and minimal elements A and {0}, respectively, and for any B, C* $\in$*sub(A),*

$$B \vee C = B + C, \quad B \wedge C = B \cap C.$$

Proposition1.2 *I(A) is a complete sublattice of sub(A).*

Definition1.1 Let f be a mapping from X into Y, and let $\mu \in F(X)$ (*F(X)* denotes the set of all fuzzy subset of X) and $\eta \in F(Y)$. Then fuzzy subsets $f(\mu)$ and $f^{-1}(\eta)$, defined by

$$f(\mu)(y) = \vee\{\mu(x) | x \in f^{-1}(y)\} \quad \forall y \in Y$$

and

$$f^{-1}(\eta)(x) = \eta(f(x)) \quad \forall x \in X,$$

are called the image of μ and the pre-image of η under f, respectively.

Definition1.2[1] Let X be a set and P(X) denotes the power set of X. A map

$$H : [0,1] \to P(X); \lambda \to H(\lambda)$$

is called a nested set of X, if

$$\lambda_1 < \lambda_2 \Rightarrow H(\lambda_1) \supseteq H(\lambda_2).$$

We denote the nested set by $H : \{H(\lambda) | \lambda \in [0,1]\}$.

If H is a nested set of X, let

$$\mu = \bigcup_{\lambda\in[0,1]} \lambda H(\lambda) \, (i.e., \mu(x) = \vee\{\lambda | x \in H(\lambda)\} \, \forall x \in X, stipulation \vee \varnothing = 0),$$

then $\mu \in F(X)$, we call this μ the fuzzy set determined by nested set H.

Proposition1.3[1] *Let H be a nested set of X,* $\mu \in F(X)$. *Then* μ *is determined by H if and only if* $\mu_{\underset{\cdot}{\lambda}} \subseteq H(\lambda) \subseteq \mu_\lambda$.

2. Important properties

Definition2.1 A fuzzy K-subalgebra(briefly, fuzzy subalgebra) of K-algebra A is a function $\mu : A \to [0,1]$ such that following properties holds:

(1) $\mu(0)=1$,

(2) $\mu(ax+by) \geq \mu(x) \wedge \mu(y)$,

(3) $\mu(xy) \geq \mu(x) \wedge \mu(y)$.

Where a,b∈K, x,y∈A.

We denote the set of all fuzzy subalgebra of A with the symbol Fsub(A).

Proposition2.1 *Let* $\mu \in F(A)$, *then* μ *is a fuzzy subalgebra of A if and only if* $\mu_\lambda (\lambda \in [0,1])$ *is a subalgebra of A.*

Proposition2.2 *Let* $\mu \in F(A)$, *then* μ *is a fuzzy subalgebra of A if and only if* $\mu_{\underset{\bullet}{\lambda}} (\lambda \in [0,1))$ *is a subalgebra of A.*

Theorem2.3 *Let* μ *be a fuzzy subset determined by nested set H. If* $H(\lambda)$ *are all subalgebra of A, for any* $\lambda \in [0,1]$, *then* μ *is a fuzzy subalgebra of A.*

Proof. For any $a,b \in K, x,y \in A$, let $\mu(x)=s$, $\mu(y)=t \geq s$, then $x \in \mu_{\underset{\bullet}{s-\varepsilon}}$, $y \in \mu_{\underset{\bullet}{t-\varepsilon}} \subseteq \mu_{\underset{\bullet}{s-\varepsilon}} (\forall \varepsilon \in (0,s))$, thus $x,y \in \mu_{\underset{\bullet}{s-\varepsilon}} \subseteq H(s-\varepsilon)$(Proposition1.3). From $H(s-\varepsilon)$ is a subalgebra we know $ax+by, xy \in H(s-\varepsilon)$, but $H(s-\varepsilon) \subseteq \mu_{s-\varepsilon}$(Proposition1.3), so $ax+by, xy \in \mu_{s-\varepsilon}$, by the arbitrary of ε, $ax+by, xy \in \mu_s$, that is $\mu(ax+by) \geq s = \mu(x) \wedge \mu(y)$, $\mu(xy) \geq s = \mu(x) \wedge \mu(y)$, and then μ is a fuzzy subalgebra of A. □

Remark2.4 *The inverse of Theorem2.3 is not right.*

For example: Let Z be the integer ring. Obviously, Z is Z-algebra. Now let S be the even number ring and define the nested set H as follows:

$$H(\lambda)=\begin{cases} S & 0 \leq \lambda < 0.5 \\ \{0,2,4\} & \lambda = 0.5 \\ \{0\} & \lambda > 0.5 \end{cases}.$$

It is easy to verify that H is a nested set of Z-algebra Z and the fuzzy subset μ determined by H is

$$\mu(k)=\vee\{\lambda | k \in H(\lambda)\}=\begin{cases} 1 & k=0 \\ 0.5 & k \in S \setminus \{0\} \\ 0 & k \notin S \end{cases}.$$

Obviously, $\mu \in Fsub(A)$, but H(0.5)={0,2,4} $\notin$ sub(A).

Definition2.2 Let μ be a fuzzy subalgebra of A. If for any $x, y \in A$, $\mu(xy) \geq \mu(x) \vee \mu(y)$, then we call μ a fuzzy algebra ideal of A.

We denote the set of all fuzzy algebra ideals with the symbol FI(A).

Theorem2.5 *Let μ be a fuzzy set determined by nested set. If $H(\lambda)(\lambda \in [0,1])$ are all algebra ideals of A, then μ is a fuzzy algebra ideal of A.*

Proof. Similar to Theorem2.3. □

Definition2.3 Let μ, η are two fuzzy set of A. We define the sum $\mu + \eta$ of μ and η as follows:

$$(\mu+\eta)(z) = \vee\{\mu(x) \wedge \eta(y) | x + y = z\}.$$

Proposition2.6 *Let $\mu, \eta \in F(A)$ are two fuzzy set of A, then*

(1) $(\mu+\eta)_\lambda \supseteq \mu_\lambda + \eta_\lambda$, (2) $(\mu+\eta)_{\underset{\cdot}{\lambda}} = \mu_{\underset{\cdot}{\lambda}} + \eta_{\underset{\cdot}{\lambda}}$.

Proof. (1) For any $z \in \mu_\lambda + \eta_\lambda$, let $z = x_0 + y_0, x_0 \in \mu_\lambda, y_0 \in \eta_\lambda$,then $(\mu+\eta)(z) = \vee\{\mu(x) \wedge \eta(y) | x + y = z\} \geq \mu(x_0) \wedge \eta(y_0) \geq \lambda$, so $z \in (\mu+\eta)_\lambda$, and then $(\mu+\eta)_\lambda \supseteq \mu_\lambda + \eta_\lambda$.

(2) The proof of $(\mu+\eta)_{\underset{\cdot}{\lambda}} \supseteq \mu_{\underset{\cdot}{\lambda}} + \eta_{\underset{\cdot}{\lambda}}$ is similar to the proof of (1); Other hand, for any $z \in (\mu+\eta)_{\underset{\cdot}{\lambda}}$, $(\mu+\eta)(z) = \vee\{\mu(x) \wedge \eta(y) | x + y = z\} > \lambda$, there some x_0, y_0, such that $z = x_0 + y_0$ and $\mu(x_0) \wedge \eta(y_0) > \lambda$, so $x_0 \in \mu_{\underset{\cdot}{\lambda}}$, $y_0 \in \eta_{\underset{\cdot}{\lambda}}$, and $z \in \mu_{\underset{\cdot}{\lambda}} + \eta_{\underset{\cdot}{\lambda}}$. Therefore (2) holds. □

Proposition2.7 *Let $\mu, \eta \in F(A)$ be two fuzzy sets determined by nested sets H, K, respectively. Then $\mu + \eta$ are determined by the nested set $H + K : \{H(\lambda) + K(\lambda) | \lambda \in [0,1]\}$.*

Proof. By the hypothesis, $\mu_{\underset{\cdot}{\lambda}} \subseteq H(\lambda) \subseteq \mu_\lambda$, $\eta_{\underset{\cdot}{\lambda}} \subseteq K(\lambda) \subseteq \eta_\lambda$, so we have $\mu_{\underset{\cdot}{\lambda}} + \eta_{\underset{\cdot}{\lambda}} \subseteq H(\lambda) + K(\lambda) \subseteq \mu_\lambda + \eta_\lambda$, and then $(\mu+\eta)_{\underset{\cdot}{\lambda}} \subseteq H(\lambda) + K(\lambda) \subseteq (\mu+\eta)_\lambda$ from Proposition2.6, this means $\mu + \eta$ is determined by the nested set $H + K : \{H(\lambda) + K(\lambda) | \lambda \in [0,1]\}$. □

Theorem2.8 *Let μ and η be two fuzzy subalgebras(ideals) of A, then $\mu + \eta$ is a fuzzy subalgebra(ideal) of A.*

Proof. We prove the Theorem to fuzzy subalgebra only. Because μ and η are two fuzzy subalgebras of A, from Proposition2.6 we have $(\mu+\eta)_{\underset{\cdot}{\lambda}} \subseteq \mu_\lambda + \eta_\lambda \subseteq (\mu+\eta)_\lambda$,this means that $\mu + \eta$ is determined by the nested set $H : \{\mu_\lambda + \eta_\lambda | \lambda \in [0,1]\}$. Since μ and η are two fuzzy subalgebras, so μ_λ and η_λ are all subalgebras, and then $H(\lambda) = \mu_\lambda + \eta_\lambda$ is a subalgebra(Proposition1.1), $\mu + \eta$ is a fuzzy subalgebra of A. □

3. Lattice structure

Theorem3.1 *The set Fsub(A) forms a complete lattice under the inclusion relation* $\subseteq$ *with the intersection as its inf. Its maximal and minimal elements are* 1_A *and* 1_0*, respectively.*

Proof. For any $\mu_i \in Fsub(A), i \in I$, where I is any nonempty index set. Then for any $k, l \in K, x, y \in A$,

$$(\underset{i\in I}{\wedge}\mu_i)(kx+ly) = \underset{i\in I}{\wedge}(\mu_i(kx+ly)) \geq \underset{i\in I}{\wedge}(\mu_i(x)\wedge\mu_i(y))$$

$$= (\underset{i\in I}{\wedge}\mu_i(x))\wedge(\underset{i\in I}{\wedge}\mu_i(y)) = (\underset{i\in I}{\wedge}\mu_i)(x)\wedge(\underset{i\in I}{\wedge}\mu_i)(y)$$

$$(\underset{i\in I}{\wedge}\mu_i)(xy) = \underset{i\in I}{\wedge}(\mu_i(xy)) \geq \underset{i\in I}{\wedge}(\mu_i(x)\wedge\mu_i(y)) = (\underset{i\in I}{\wedge}\mu_i)(x)\wedge(\underset{i\in I}{\wedge}\mu_i)(y).$$

Hence, we conclude that $\underset{i\in I}{\wedge}\mu_i \in Fsub(A)$. Obviously, $1_A \in Fsub(A)$, thus we can assert that Fsub(A) forms a completee lattice under the order $\subseteq$. Other conclusion of this theorem is easy. □

In the lattice Fsub(A), $\vee,\wedge$ denote the sup, inf, respectiveely.

Proposition3.2 *FI(A) is the complete sublattice of Fsub(A).*

Proof is easy and omitted. □

Theorem3.3 *Let* $\mu, \eta \in Fsub(A)$*, then* $\mu\vee\eta = \mu+\eta$.

Proof. By Theorem2.8, $\mu+\eta \in Fsub(A)$. Since $\mu_\lambda, \eta_\lambda \subseteq \mu_\lambda + \eta_\lambda \subseteq (\mu+\eta)_\lambda$, and then $\mu, \eta \subseteq \mu+\eta$. If $\xi \in Fsub(A)$ and $\mu, \eta \subseteq \xi$, then $\mu_\lambda, \eta_\lambda \subseteq \xi_\lambda$, and then $\mu_\lambda + \eta_\lambda \subseteq \xi_\lambda$, but $(\mu+\eta)_{\underset{\cdot}{\lambda}} \subseteq \mu_\lambda + \eta_\lambda$ (proposition2.6), thus $(\mu+\eta)_{\underset{\cdot}{\lambda}} \subseteq \xi_\lambda$, so $\mu+\eta \subseteq \xi$, this means that $\mu+\eta$ is a minimal fuzzy subalgebra of A which contains μ and η, that is $\mu\vee\eta = \mu+\eta$. □

Remark3.4 *The lattice Fsub(A) is not distributive.*

Proof. Suppose, if possible, Fsub(A) is distributive. Let A be a ring, Z the ring of integers, then A is a Z-algebra. In this case, K=Z, Fsub(A) is the lattice of all fuzzy subrings and is distributive, of course, the lattice of all fuzzy ideals of a ring A is also distributivity. This contradicts the Theorem(The lattice of all fuzzy ideals of a ring is not distributivity)[5]. □

Theorem3.5 *The lattice Fsub(A) is modular.*

Proof. For any $\mu, \eta \in Fsub(A)$, it's easy to verify that $(\mu\wedge\eta)_\lambda = \mu_\lambda \wedge \eta_\lambda$, $(\mu\wedge\eta)_{\underset{\cdot}{\lambda}} = \mu_{\underset{\cdot}{\lambda}} \wedge \eta_{\underset{\cdot}{\lambda}}$.

For any $\mu, \eta, \gamma \in Fsub(A)$ and $\eta \supseteq \mu$, we will prove $\mu\wedge(\eta\vee\gamma) = \eta\vee(\mu\wedge\gamma)$. From Theorem 3.3 and Proposition2.6 we have $\mu\wedge(\eta\vee\gamma) = \mu\wedge(\eta+\gamma)$ and

$$(\mu\wedge(\eta+\gamma))_{\underset{\cdot}{\lambda}} = \mu_{\underset{\cdot}{\lambda}} \wedge (\eta+\gamma)_{\underset{\cdot}{\lambda}} \subseteq \mu_\lambda \wedge (\eta_\lambda + \gamma_\lambda) \subseteq \mu_\lambda \wedge (\eta+\gamma)_\lambda = (\mu\wedge(\eta+\gamma))_\lambda,$$

$$(\eta+(\mu\wedge\gamma))_{\underset{\cdot}{\lambda}} \subseteq \eta_\lambda + (\mu\wedge\gamma)_\lambda = \eta_\lambda + (\mu_\lambda\wedge\gamma_\lambda) \subseteq (\eta+(\mu\wedge\gamma))_\lambda.$$

This shows that $\mu \wedge (\eta + \gamma)$ and $\eta + (\mu \wedge \gamma)$ are determined by the nested sets $\mu_\lambda \wedge (\eta_\lambda + \gamma_\lambda)$ and $\eta_\lambda + (\mu_\lambda \wedge \gamma_\lambda))_\lambda$, respectively. But $\mu_\lambda, \eta_\lambda$ and γ_λ are all the crisp K-subalgebras of A and $\mu_\lambda \supseteq \eta_\lambda$, by Proposition1.1 we have $\mu_\lambda \wedge (\eta_\lambda + \gamma_\lambda) = \eta_\lambda + (\mu_\lambda \wedge \gamma_\lambda)$, and then $\mu \wedge (\eta \vee \gamma) = \eta \vee (\mu \wedge \gamma)$. □

Corollary3.6 *The lattice FI(A) is modular.*

Proposition3.7 *If* $B, C \in Fsub(A)$ *and* $B \subset C$. *Let*

$$Fsub_{mid}(B,C) = \{D | D \in Fsub(A) \text{ and } B \subseteq D \subseteq C\}.$$

Then $Fsub_{mid}(B,C)$ *is the complete sublattice of Fsub(A) with maximal and minimal elements B and C, respectively. Of course,* $Fsub_{mid}(B,C)$ *is modular.*

Proof. Omitted. □

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