$(\in, \in \vee q)$-Fuzzy algebras

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Abstrct: In this paper, the concept of $(\in, \in \vee q)$-fuzzy fields and $(\in, \in \vee q)$-fuzzy algebra are introduced. A necessary and sufficient condition for a fuzzy sets to be an $(\in, \in \vee q)$-fuzzy algebra is stated, and using the extension principle of fuzzy sets, images and inverse-images of an $(\in, \in \vee q)$-fuzzy algebra under a algebraic homomorphism are studied.

keywords: Fuzzy algebra; belongs to; quasi-coincident; $(\in, \in \vee q)$-fuzzy field; $(\in, \in \vee q)$-fuzzy algebra.

1. Introduction

In 1996, S. K. Bhakat and P. Das [4-5] used relation of "belongs to" and "quasi-coincident" between fuzzy point and fuzzy set, introduced the concepts of an $(\in, \in \vee q)$-fuzzy subgroup and $(\in, \in \vee q)$-fuzzy subrings, and obtained some fundamental results pertaining to these notions. Fuzzy field and Fuzzy algebra over fuzzy field were researched by Nada [1], Gu wenxiang and Lu Tu [2] and Dang [3]. In this paper, $(\in, \in \vee q)$-fuzzy field and $(\in, \in \vee q)$-fuzzy algebra over $(\in, \in \vee q)$-fuzzy field are defined, and their some properties are studied.

2. Preliminaries

Let X be any non-empty set.

Definition 2.1. A map $\lambda: X \to [0,1]$ is called a fuzzy set of X.

Definition 2.2. (Ming and Ming [6]). A fuzzy set λ of X of the from

$$\lambda(y)=\begin{cases} t(\neq 0) & \text{if} \quad y=x, \\ 0 & \text{if} \quad y\neq x. \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Definition 2. 3. (Ming and Ming [6]). A fuzzy point x_t is said to belong to (resp. be quasi - coincident with) a fuzzy set λ, written as $x_t \in \lambda$ (resp. $x_t q \lambda$) if $\lambda(x) \geqslant t$ (resp. $\lambda(x)+t>1$). If $x_t \in \lambda$ or $x_t q \lambda$, then we write $x_t \in \vee q \lambda$.

For any $t, r \in [0,1]$, $M(t,r)$ will denote $\min(t,r)$. $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

Definition 2. 4. (Gu and Lu [2]). Lex X be a field and F a fuzzy set of X. If the following conditions hold:

(Ⅰ) $F(x+y) \geqslant F(x) \wedge F(y), x, y \in X$;

(Ⅱ) $F(-x) \geqslant F(x), x \in X$;

(Ⅲ) $F(xy) \geqslant F(x) \wedge F(y), x, y \in X$;

(Ⅳ) $F(x^{-1}) \geqslant F(x), x(\neq 0) \in X$.

We call F a fuzzy field of X.

Definition 2. 5. (Gu. andLu. [2]) Let F be a fuzzy field of the field X. Y a algebra over X and A a fuzzy set of Y. Suppose the following conditions holds:

(Ⅰ) $A(x+y) \geqslant A(x) \wedge A(y), x, y \in Y$;

(Ⅱ) $A(\lambda x) \geqslant F(\lambda) \wedge A(x), \lambda \in X$ and $y \in Y$;

(Ⅲ) $A(xy) \geqslant A(x) \wedge A(y), x, y \in Y$;

(Ⅳ) $F(1) \geqslant A(x), x \in Y$.

Then we call A a fuzzy algebra over fuzzy field F.

3. $(\in, \in \vee q)$- fuzzy field

Definition 3. 1. Let X be a field and F a fuzzy set of X. If for all $x, y \in X$ and $t, r \in (0,1]$, the following conditions hold:

(Ⅰ) $x_t, y_t \in F \Rightarrow (x+y)_{M(t,r)} \in \vee q F$;

(Ⅱ) $x_t \in F \Rightarrow (-x)_t \in \vee q F$;

(Ⅲ) $x_t, y_r \in F \Rightarrow (xy)_{M(t,r)} \in \vee qF$;

(Ⅳ) $x_t \in F\ (x \neq 0) \Rightarrow (x^{-1})_t \in \vee qF$.

we call F an $(\in, \in \vee q)$-fuzzy field of X.

Remark 3.1. The condition(Ⅰ) in Definition 3.1 is equivalent to

(Ⅰ′) $F(x+y) \geqslant M(F(x), F(y), 0.5), x, y \in X$;

The condition (Ⅱ) of Definition 3.1 is equivalent to

(Ⅱ′) $F(-x) \geqslant M(f(x), 0.5), x \in X$;

The condition (Ⅲ) of Definition 3.1 is equivalent to

(Ⅲ′) $F(xy) \geqslant M(F(x), F(y), 0.5), x, y \in X$;

The condition(Ⅳ) of Definition 3.1 is equivalent to

(Ⅳ′) $F(x^{-1}) \geqslant M(F(x), 0.5), x(\neq 0) \in X$.

Remark 3.2. we note that if F is a fuzzy field of X according to be Definition 2.4, then F is an $(\in, \in \vee q)$-fuzzy field of X, but the converse is not true.

proposition 3.1. If F is an $(\in, \in \vee q)$-fuzzy field of X, then

(Ⅰ) $F(0) \geqslant M(F(x), 0.5), x \in X$;

(Ⅱ) $F(1) \geqslant M(F(x), 0.5), x \in X$.

proposition 3.2. Let X and Y be fields and f a homomorphism of X into Y. Suppose that F is an $(\in, \in \vee q)$-fuzzy field of X and G is an $(\in, \in \vee q)$-fuzzy field of Y. Then

(Ⅰ) f(F) is an $(\in, \in \vee q)$-fuzzy field of Y.

(Ⅱ) $f^{-1}(G)$ is an $(\in, \in \vee q)$-fuzzy field of X.

4. $(\in, \in \vee q)$-fuzzy algebra over $(\in, \in \vee q)$-fuzzy field

Definition 4.1. Let F be an $(\in, \in \vee q)$-fuzzy field of the field X, Y a algebra over X and A a fuzzy set of Y. If for all $x, y \in Y$, $\lambda \in X$ and $t, \gamma \in (0,1]$, the following conditions hold:

(Ⅰ) $x_t, y_r \in A \Rightarrow (x+y)_{M(t,r)} \in \vee qA$;

(Ⅱ) $x_t \in A, \lambda_r \in F \Rightarrow (\lambda x)_{M(t,r)} \in \vee qA$;

(Ⅲ) $x_t \in A, y_r \in A \Rightarrow (xy)_{M(t,r)} \in \vee qA$;

(Ⅳ) $F(1) \geqslant M(A(x), 0.5)$.

Then we call A an $(\in, \in \vee q)$- fuzzy algebra over $(\in, \in \vee q)$- fuzzy field F.

Remark 4.1. The condition(I) in definition 4.1 is equivalent to

(I′) $A(x+y) \geqslant M(A(x), A(y), 0.5), x, y \in Y$;

The condition (II) of Definition4.1 is equivalent to

(II′) $A(\lambda x) \geqslant M(F(\lambda), A(x), 0.5), \lambda \in X, x \in Y$;

The condition (III) of Definition4.1 is equivalent to

(III′) $A(xy) \geqslant M(A(x), A(y), 0.5), X, y \in Y$;

proof. It is only proved that (II) is equivalent to (II′).

(II)⇒(II′): Let $\lambda \in X, x \in Y$. Let $M(F(\lambda), A(x)) < 0.5$. Assume that $A(\lambda x) < M(F(\lambda), A(x))$. Choose t such that $A(\lambda x) < t < M(F(\lambda), A(x))$. The $\lambda_t \in F, x_t \in A$ but $(\lambda x)_t \overline{\in \vee q}$ A which contradicts(II). So $A(\lambda x) \geqslant M(F(\lambda), A(x))$. Next, Let $M(F(\lambda), A(x)) \geqslant 0.5$. Assume that $A(\lambda x) < 0.5$. Then $\lambda_{0.5} \in F, x_{0.5} \in A$, but $(\lambda x)_{0.5} \overline{\in \vee q}$ A, a contradiction. Hence (II′) holds.

(II′)⇒(II). Let $\lambda_r \in F, x_t \in A$. Then $F(\lambda) \geqslant r$. $A(x) \geqslant t$. By (II′), $A(\lambda x) \geqslant M(F(\lambda)), A(x), 0.5) \geqslant M(r, t, 0.5)$. Thus, $A(\lambda x) \geqslant M(r, t)$, if r or $t \leqslant 0.5$ and $A(\lambda x) \geqslant 0.5$, if $r, t > 0.5$. Hence, $(\lambda x)_{M(t,r)} \in \vee qA$.

Remark 4.2. We note that if A is a fuzzy algebra over fuzzy field F according to be Definition2.5, then A is an $(\in, \in \vee q)$- fuzzy algebra over $(\in, \in \vee q)$- fuzzy field F according to the Definition 4.1. But the converse is not trus.

proposition 4.1. If A is an $(\in, \in \vee q)$-fuzzy algebra over an $(\in, \in \vee q)$-fuzzy field F, then $F(0) \geqslant M(A(x), 0.5)$.

Proposition 4.2. Let F be an $(\in, \in \vee q)$- fuzzy field of the field X, Y a algebra over X and A a fuzzy set of Y. Then A is an $(\in, \in \vee q)$- fuzzy algebra over an $(\in, \in \vee q)$- fuzzy field F iff

(I)for any $\lambda, \mu \in X$ and $x, y \in Y$;

$A(\lambda x + \mu y) \geqslant M(F(\lambda), F(\mu), A(x), A(y), 0.5)$;

(II)for any $x, y \in Y$,

$A(xy) \geqslant M(A(x), A(y), 0.5)$;

(Ⅲ) $F(1) \geqslant M(A(x), 0.5)$.

Proposition 4.3. Let Y and Z be algebras over the field X, f a algebraic homomorphism of Y into Z and A a fuzzy set of Z. If A an $(\in, \in \vee q)$-fuzzy algebra over an $(\in, \in \vee q)$-fuzzy field F. Then $f^{-1}(A)$ is an $(\in, \in \vee q)$-fuzzy algebra over $(\in, \in \vee q)$-fuzzy field F.

Proof. (Ⅰ) For any $\lambda, \mu \in X$ and $x, y \in Y$,

$$f^{-1}(A)(\lambda x+\mu y)=A(f(\lambda x+\mu y))$$
$$=A(\lambda f(x)+\mu f(y)) \geqslant M(F(\lambda), A(f(x)), F(\mu), A(f(y)), 0.5)$$
$$=M(F(\lambda), f^{-1}(A)(x), F(\mu), f^{-1}(A)(y), 0.5)$$

(Ⅱ) For any $x, y \in Y$,

$$f^{-1}(A)(xy)=A(f(xy))=A(f(x)f(y))$$
$$\geqslant M(A(f(x)), A(f(y)), 0.5)$$
$$=M(f^{-1}(A)(x), f^{-1}(A)(y), 0.5$$

(Ⅲ) For any $x \in Y$,

$$F(1) \geqslant M(A(f(x)), 0.5)=M(f^{-1}(A)(x), 0.5)$$

Hence $f^{-1}(A)$ is an $(\in, \in \vee q)$-fuzzy algebra over $(\in, \in \vee q)$-fuzzy field F.

Proposition 4.4. Let Y and Z be algebras over the field X, f an algebraic homomorphism of Y into Z and A a fuzzy set of Y. If A an $(\in, \in \vee q)$-fuzzy alegbra over $(\in, \in \vee q)$-fuzzy field F. Then $f(A)$ is an $(\in, \in \vee q)$-fuzzy algebra over $(\in, \in \vee q)$-fuzzy field F.

Proof. (Ⅰ) For any $x, y \in Z$.

$$f(A)(x+y)=\sup_{f(z)=x+y} A(z) \geqslant \sup_{\substack{f(\bar{x})=x \\ f(\bar{y})=y}} A(\bar{x}+\bar{y})$$
$$\geqslant \sup_{\substack{f(\bar{x})=x \\ f(\bar{y})=y}} M(A(\bar{x}), A(\bar{y}), 0.5)$$
$$=M\left(\sup_{f(\bar{x})=x} A(\bar{x}), \sup_{f(\bar{y})=y} A(\bar{y}), 0.5\right)$$
$$=M(f(A)(x), f(A)(y), 0.5).$$

(Ⅱ) For all $x \in Z$ and $\lambda \in X$,

$$f(A)(\lambda x)=\sup_{f(z)=\lambda x} A(z)=\sup_{f(z)=x} A(\lambda z)$$
$$\geqslant \sup_{f(z)=x} M(F(\lambda), A(z), 0.5)$$
$$=M(F(\lambda), \sup_{f(z)=x} A(z), 0.5)$$

$$=M(F(\lambda),f(A)(x),0.5)。$$

(Ⅲ)For all $x,y\in Z$,

$$f(A)(xy)=\sup_{f(z)=xy}A(z)\geqslant\sup_{\substack{f(\bar{x})=x\\ f(\bar{y})=y}}A(\bar{x}\,\bar{y})$$

$$\geqslant\sup_{\substack{f(\bar{x})=x\\ f(\bar{y})=y}}M(A(\bar{x}),A(\bar{y}),0.5)$$

$$=M(\sup_{f(\bar{x})=x}A(\bar{x}),\sup_{f(\bar{y})=y}A(\bar{y}),0.5)$$

$$=M(f(A)(x),f(A)(y),0.5).$$

(Ⅳ). For all $z\in Y$, $F(1)\geqslant M(A(z),0.5)$. Hence, for all $x\in Z$,

$$F(1)\geqslant\sup_{f(z)=x}M(A(z),0.5)=M(\sup_{f(z)=x}A(z),0.5)$$

$$=M(f(A)(x),0.5).$$

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