Some Properties of minimum values for the convex fuzzy valued mappings*

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Abstract In this paper, on the basis of the concepts introduced by S. Nanda of convex fuzzy valued mappings and the multiplication operation of fuzzy numbers concerning real numbers, we prove that fuzzy numbers satisfy distributive law with respect to the addition of positive real numbers. Furthermore, on a general open convex set $\Omega \subset R^n$, we obtain that the local minimum values of convex fuzzy valued mappings are equivalent to the global minimum values of theirs.

Keywords Fuzzy numbers, convex fuzzy valued mappings, local minimum values, global minimum value.

1 Introduction

The concept of convex fuzzy sets were originally introduced by L. A. Zadeh[7]. Subsequently a lot of scholars did a great deal of work at the aspects of their theories and applications. Some properties of convex fuzzy sets were studied and given by Lowen[2], Katsaras and Liu[3], Brown [4]. The concept of the convex fuzzy mappings has first been introduced and some results including some applications to nondifferentiable optimization have been investigated by Nanda[1]. In recent years, as the studies deepen constantly, the theories of convex fuzzy valued mappings are being applied step by step to some fields of Mathematics.

For simplicity, we consider only the convex fuzzy valued mappings defined on the Euclidean space R^n. In the paper, we will study the equivalences problem of the local minimum values and the global minimum value of the convex fuzzy valued mappings on an open convex set $\Omega \subset R^n$.

2 Preliminaries

Let $[H] = \{\bar{a} = [a^-, a^+] : a^- \leqslant a^+, a^-, a^+ \in R\}$ denote the set of all interval numbers on the real number field R.

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Let $\bar{a},\bar{b},\bar{c} \in [H]$, where $\bar{a} = [a^-,a^+], \bar{b} = [b^-,b^+], \bar{c} = [c^-,c^+]$. we define

(1) $\bar{a} = \bar{b}$ iff $a^- = b^-$ and $a^+ = b^+$

(2) $\bar{a} \leqslant \bar{b}$ iff $a^- \leqslant b^-$ and $a^+ \leqslant b^+$

(3) $\bar{a} < \bar{b}$ iff $\bar{a} \leqslant \bar{b}$ and $\bar{a} \neq \bar{b}$

(4) $\bar{a} + \bar{b} = \bar{c}$ iff $a^- + b^- = c^-$ and $a^+ + b^+ = c^+$

(5) $k\bar{a} = [ka^-, ka^+]$, *whenever* $k \geqslant 0$

For $\bar{a},\bar{b} \in [H]$, their distance is defined as

$$d(\bar{a},\bar{b}) = \max\{|a^- - b^-|, |a^+ - b^+|\}$$

Obviously, $([H],d)$ is a metric space, where d is an Hausdorff distance, and $([H],\leqslant)$ constitutes a partial order set.

Let P and Q be classical sets, we define the multiplication of a set concerning a real number and the sum of two sets as follows.

(1) $kP = \{kx \mid x \in P\}$, *whenever* $k \in R$,

(2) $P + Q = \{x + y \mid x \in P, y \in Q\}$

Definition 2.1 Let a mapping $A: R \to [0,1]$. Then A is called a fuzzy number on R, if the following conditions are satisfied.

(1) A is normal, i.e., there exists $x_0 \in R$ such that $A(x_0) = 1$.

(2) A is a convex fuzzy set, i.e., for arbitrary $x,y \in R, \lambda \in [0,1]$, $A(\lambda x + (1-\lambda)y) \geqslant A(x) \wedge A(y)$ holds.

(3) A is upper semi-continuous.

(4) Let support $\mathrm{supp}A = \{x \in R : A(x) > 0\}$, its closure $\overline{\mathrm{supp}A}$ is compact.

Denote the family of all fuzzy numbers on R as $F^*(R)$.

By above definition, if $A \in F^*(R)$, for every $\alpha \in [0,1]$, we are easy to know its cut-set $A_\alpha = \{x \in R : A(x) \geqslant \alpha\}$ is a closed interval, denote it as $A_\alpha = [A_\alpha^-, A_\alpha^+]$

Let a mapping $\bar{\rho}: F^*(R) \times F^*(R) \to [0, +\infty)$

$$\bar{\rho}(A,B) \triangleq \sup_{0 \leqslant \alpha \leqslant 1} d(A_\alpha, B_\alpha)$$

We can verify that $\bar{\rho}$ is also a distance on $F^*(R)$

Let $A,B,C \in F^*(R)$, we define

$A \leqslant B$ iff $A_\alpha \leqslant B_\alpha$ for any $\alpha \in [0,1]$

$A + B = C$, where $A_\alpha + B_\alpha = C_\alpha$, for any $\alpha \in [0,1]$.

Definition 2.2 Let $A \in F^*(R)$, for all $k \geqslant 0, x \in R$, define

$$(kA)(x) = \begin{cases} A(\frac{x}{k}) & , \quad \text{if} \quad k \neq 0 \\ 1 & , \quad \text{for } x = 0 \quad \text{if} \quad k = 0 \\ 0 & , \quad \text{for } x \neq 0 \quad \text{if} \quad k = 0 \end{cases}$$

From this definion, obviously, we have $1A = A$ and $0A = \tilde{0}$,

where $\tilde{0}(x) = \begin{cases} 1, & \textit{if} \quad x = 0 \\ 0, & \textit{if} \quad x \neq 0 \end{cases}$

3 Main results

In the course of applications, as if in classical real valued functions, we are interested in the global properties of the extremum, i.e., in the whole field of definitions Ω and the local neighbourhood $U(x_0,\delta)$ regarding x_0 as a centre, δ as a radius, whether we have $f(x) \geqslant f(x_0)$, for all $x \in \Omega$ iff $f(x) \geqslant f(x_0)$, for all $x \in U(x_0,\delta)$? In the ordinary case, it is not certain. But as for convex fuzzy valued mappings, the following several theorem tell us that the global minimum value and the local minimum values are equivalent.

Throughout this section, we let Ω be a convex set in the n-dimensional Euclidean space R^n.

Definition 3.1 Let a mapping $f:\Omega \to F^*(R)$. Then f is called a convex fuzzy valued mapping on Ω, if for arbitrary $x,y \in \Omega, \lambda \in [0,1], f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y)$ always holds.

Especially, whenever $0 < \lambda < 1$, and above strict inequality holds, then f is called a strictly convex fuzzy valued mapping.

Similarly, by $f(\lambda x + (1-\lambda)y) \geqslant \lambda f(x) + (1-\lambda)f(y)$, we may define the concave fuzzy valued mappings, without loss of generality, we only study convex fuzzy valued mappings.

Definition 3.2 Let convex set $\Omega \subset R^n$, a mapping $f:\Omega \to F^*(R)$, if there exists a neighbourhood $U(x_0,\delta) \subset \Omega$ regarding $x_0 \in \Omega$ as a centre, δ as a radius such that $f(x) \geqslant f(x_0)$, for all $x \in U(x_0,\delta)$. Then x_0 is called a local minimum point of a fuzzy valued mapping f, $f(x_0)$ is called a local minimum value of f. Where $U(x_0,\delta) = \{x \in \Omega : \| x - x_0 \| < \delta\}$, and $\| \cdot \|$ is a norm in R^n.

Similarly, if there exists a point $x_0 \in \Omega$ such that $f(x) \geqslant f(x_0)$, for any $x \in \Omega$. Then x_0 is called a global minimum point of a fuzzy valued mapping f, $f(x_0)$ is called a global minimum value of f.

In order to rear applications, we will first give some important properties with respect to fuzzy numbers, i.e., Theorem 3.1-3.3.

Theorem 3.1 If $A \in F^*(R)$, for any $k > 0$, Then $(kA)\alpha = kA_\alpha$, for all $\alpha \in [0,1]$.

Proof By definition 2.2 and the definition of the cut-sets of fuzzy numbers.

Evidently, for all $x \in (kA)_\alpha$ iff $(kA)(x) \geqslant \alpha$ iff $A(\frac{x}{k}) \geqslant \alpha$ iff $\frac{x}{k} \in A_\alpha$ iff $x \in kA_\alpha$.

Thus, the proof is completed.

Theorem 3.2 If $A,B \in F^*(R)$ and $A \leqslant B$, whenever $k > 0$. Then $kA \leqslant kB$.

Proof By the definition of order "$\leqslant$" and Theorem 3.1, We need only prove $kA_\alpha \leqslant kB_\alpha$, for all $\alpha \in [0,1]$. From the known conditions, it is clear.

Theorem 3.3 If $A \in F^*(R)$, for any $k_1 > 0, k_2 > 0$. Then $(k_1 + k_2)A = k_1A + k_2A$. i.e., fuzzy number A concerning the addition of positive real numbers satisfy distributive law.

Proof By decomposition theorem of fuzzy sets and Theorem 3.1, we need only prove that

$$(k_1 + k_2)A_\alpha = k_1A_\alpha + k_2A_\alpha \text{, for all } \alpha \in [0,1].$$

In fact, for every $x \in (k_1 + k_2)A_\alpha, \alpha \in [0,1]$, there exists a point $x_0 \in A_\alpha$ such that $x = (k_1 + k_2)x_0$. Therefore, $x = k_1x_0 + k_2x_0 \in k_1A_\alpha + k_2A_\alpha$ is obvious,

$$\text{i.e., } (k_1 + k_2)A_\alpha \subset k_1A_\alpha + k_2A_\alpha .$$

On the contrary, for every $x \in k_1A_\alpha + k_2A_\alpha$, there exists points $x_0 \in A_\alpha$ and $y_0 \in A_\alpha$ such that $x = k_1x_0 + k_2y_0$ and $A(x_0) \geqslant \alpha, A(y_0) \geqslant \alpha$.

On the other hand, as $A \in F^*(R)$, from Definition 2.1, we know that A is certain a convex fuzzy set. Hence, we have

$$\begin{aligned} A\left(\frac{x}{k_1 + k_2}\right) &= A\left(\frac{k_1x_0 + k_2y_0}{k_1 + k_2}\right) \\ &= A\left(\frac{k_1}{k_1 + k_2}x_0 + \left(1 - \frac{k_1}{k_1 + k_2}\right)y_0\right), \quad \text{where} \quad \frac{k_1}{k_1 + k_2} \in (0,1) \\ &\geqslant A(x_0) \wedge A(y_0) \\ &\geqslant \alpha \wedge \alpha \\ &= \alpha \end{aligned}$$

Consequently, $\frac{x}{k_1 + k_2} \in A_\alpha$ or $x \in (k_1 + k_2)A_\alpha$

i.e., $k_1A_\alpha + k_2A_\alpha \subset (k_1 + k_2)A_\alpha$

Thus, the proof is finished.

The following several Theorems are main results in this paper.

Theorem 3.4 Let f be a convex fuzzy valued mapping defined on an open convex set $\Omega \subset R^n$. Then the set of the points which make f get a global minimum value is a convex set.

Proof First, we will prove a generality conclusion.

Take arbitrary $A \in F^*(R)$, define set $m(f,A) = \{x \in \Omega : f(x) \leqslant A\}$.

We can prove straightforward that $m(f,A)$ is a convex set.

Actually, letting $x_1, x_2 \in m(f,A)$, then $f(x_1) \leqslant A, f(x_2) \leqslant A$.

Consequently, for any $0 < \lambda < 1$, from Theorem 3.1 − 3.3, We have

$$\begin{aligned} f(\lambda x_1 + (1 - \lambda)x_2) &\leqslant \lambda f(x_1) + (1 - \lambda)f(x_2) \\ &\leqslant \lambda A + (1 - \lambda)A \\ &= (\lambda + 1 - \lambda)A \\ &= A \end{aligned}$$

Thus, $\lambda x_1 + (1-\lambda)x_2 \in m(f,A)$

i. e., $m(f,A)$ is a convex set on Ω.

Now, let x_0 be a global minimum point of f on Ω, writing $f(x_0) = B \in F^*(R)$ Therefore, $m(f,B) = \{x \in \Omega : f(x) \leqslant B\}$. And so $m(f,B)$ is a set of all points which make f get a minimum value on Ω.

According to above generality conclusion, we obtain that $m(f,B)$ is also a convex set. Consequently, the proof is completed.

Theorem 3.5 If f is a convex fuzzy valued mapping defined on an open convex set $\Omega \subset R^n$. Then an arbitrary local minimum point of f must be a global minimum point of f on Ω.

Proof Let x_0 be a local minimum point of f on Ω, i. e., there exists a δ-neighbourhood $U(x_0, \delta)$ of x_0 such that $f(x_0) \leqslant f(x)$, for arbitrary $x \in U(x_0,\delta)$

Let $x \in \Omega - U(x_0,\delta)$, Clearly, we have $x \neq x_0$ and $x \notin U(x_0,\delta)$,

i. e., $\| x - x_0 \| \geqslant \delta$ or $\frac{\delta}{\| x - x_0 \|} \leqslant 1$ (where $\| x - x_0 \| \neq 0$)

We choose λ: $0 < \lambda < \frac{\delta}{\| x - x_0 \|}$

Then $\| ((1-\lambda)x_0 + \lambda x) - x_0 \| = \lambda \| x - x_0 \| < \delta$

i. e., $(1-\lambda)x_0 + \lambda x \in U(x_0,\delta)$.

As x_0 is a local minimum point, we can derive from that

$$f(x_0) \leqslant f((1-\lambda)x_0 + \lambda x) \leqslant (1-\lambda)f(x_0) + \lambda f(x)$$

According to the operation properties of fuzzy numbers and Theorem 3.1 – 3.3, we obtain

For all $\alpha \in [0,1]$

$$(f(x_0))_\alpha \leqslant ((1-\lambda)f(x_0) + \lambda f(x))_\alpha$$
$$= (1-\lambda)(f(x_0))_\alpha + \lambda(f(x))_\alpha$$

or $[f_\alpha^-(x_0), f_\alpha^+(x_0)] \leqslant [(1-\lambda)f_\alpha^-(x_0) + \lambda f_\alpha^-(x), (1-\lambda)f_\alpha^+(x_0) + \lambda f_\alpha^+(x)]$

Thus, we derive from that $\begin{cases} f_\alpha^-(x_0) \leqslant (1-\lambda)f_\alpha^-(x_0) + \lambda f_\alpha^-(x) \\ f_\alpha^+(x_0) \leqslant (1-\lambda)f_\alpha^+(x_0) + \lambda f_\alpha^+(x) \end{cases}$

Therefore, we have

$$\begin{cases} f_\alpha^-(x_0) \leqslant f_\alpha^-(x) \\ f_\alpha^+(x_0) \leqslant f_\alpha^+(x) \end{cases}, \text{ for all } x \in \Omega - U(x_0,\delta)$$

Furthermore, $(f(x_0))_\alpha = [f_\alpha^-(x_0), f_\alpha^+(x_0)] \leqslant [f_\alpha^-(x), f_\alpha^+(x)] = (f(x))_\alpha$

Taking advantage of the decomposition theorem of fuzzy sets, we get

$$f(x_0) \leqslant f(x), \quad \text{for all} \quad x \in \Omega - U(x_0,\delta)$$

Finally, we obtain $f(x_0) \leqslant f(x)$, for all $x \in \Omega$

Hence, the proof is completed

Theorem 3.6 Let f be a strictly convex fuzzy valued mapping defined on an open convex set $\Omega \subset R^n$, if f has a global minimum value on Ω. Then it is gotten at an unique point on Ω.

Proof Let $A \in F^*(R)$ be a global minimum value of f on Ω . i.e., there exists $x_0 \in \Omega$ such that $A = f(x_0) \leqslant f(x)$, for arbitrary $x \in \Omega$.

Let $m(f = A) = \{x \in \Omega : f(x) = A\}$

Then by Theorem 3.4, we know that $m(f = A)$ is a convex set.

Suppose there exists $x_1 \in \Omega$ and $x_1 \neq x_0$ such that $f(x_0) = f(x_1) = A$

Obviously, we have $x_0, x_1 \in m(f = A)$.

And so, whenever $0 < \lambda < 1$, we have

$$\lambda x_0 + (1 - \lambda)x_1 \in m(f = A)$$

Hence $$f(\lambda x_0 + (1 - \lambda)x_1) = A \qquad (*)$$

On the other hand, as f is a strictly convex fuzzy valued mapping, we obtain that

$$\begin{aligned} f(\lambda x_0 + (1 - \lambda)x_1) &< \lambda f(x_0) + (1 - \lambda)f(x_1) \\ &= \lambda A + (1 - \lambda)A \\ &= (\lambda + 1 - \lambda)A \\ &= A \end{aligned}$$

This contradicts (*)

Therefore, f can be only gotten an unique global minimum value on Ω .

The theorem is proved.

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