GENERATED T-NORMS AND THE CONVERGENCE OF GENERATORS

JÁN RYBÁRIK

ABSTRACT. Paper extends the results of Jenei and Mesiar on the convergent additive (multiplicative) generators yielding convergent triangular norms. We are not restricted to continuous or Archimedean t-norms as in mentioned approaches. The pointwise convergence of additive generators to a limit additive generator yields the convergence of corresponding t-norms.

1. INTRODUCTION.

Recently, Jenei [1] and Mesiar [11] have shown that the pointwise convergence $\lim_{n\to\infty} f_n(x) = f(x), \quad x \in [0,\ 1]$ of some continuous additive generators $(f_n)_{n\in N}$ to a continuous additive generator f (up to the point 0, possibly) ensures the pointwise convergence $\lim_{n\to\infty} T_n(x,y) = T(x,y), \quad (x,y) \in [0,\ 1]^2$ of the corresponding t-norms. These t-norms are then obviously continuous and Archimedean. For more details and terminology concerning triangular norms we recommend [3,12]. More, this convergence is uniform, see [2] or [6]. However, not only continuous Archimedean t-norms are generated by means of additive (multiplicative) generators. Non-continuous additive generators are discussed, for example, in [4,13,14,15]. Therefore it is natural to investigate the limit properties of generated t-norms based on the convergence of corresponding additive (multiplicative) generators without the limitation to continuous Archimedean t-norms. Note that because of the one-to-one correspondence between additive generators and multiplicative generators we will deal in this paper with additive generators only.

2. GENERATED T-NORMS.

Definition 1 [10]. Let $f : [0,\ 1] \to [0,\ \infty]$ be a strictly decreasing mapping such that $f(1) = 0$. Then f is called **a conjunctive additive generator**.

Recall that for any conjunctive additive generator f we can define its pseudo-inverse
$f^{(-1)} : [0,\ \infty] \to [0,\ 1]$ by $f^{(-1)}(x) = \sup\{\, t \in [0,\ 1] \mid f(t) > x \,\}$.
Pseudo-inverse $f^{(-1)}$ of a conjunctive additive generator is always a continuous non-decreasing mapping, see [5]. Immediatetely we get the next result.

1991 *Mathematics Subject Classification.* 0521.
Key words and phrases. additive generator, triangular norm.

Proposition 1. *Let* $f : [0, 1] \to [0, \infty]$ *be a conjunctive additive generator. Then the mapping* $T_f : [0, 1]^2 \to [0, \infty]$,

$$T_f(x,y) = f^{(-1)}(f(x) + f(y))$$

is a commutative, non-decreasing extension of the classical Boolean conjunction. Moreover, the element 1 is a neutral element of T_f .

Evidently T_f is a t-norm if and only if it is an associative operator on $[0, 1]^2$.

Definition 2. Let $f : [0, 1] \to [0, \infty]$ be a conjunctive additive generator such that T_f is associative and hence a t-norm. Then f will be called **an additive generator** (of a t-norm T_f). The class of all additive generators (of some t-norm) we will denote by $\mathcal{F}$.

Note that till now the class $\mathcal{F}$ was not fully characterized. Some sufficient conditions for $f \in \mathcal{F}$ can be found in [5,6,15] while some necessary conditions are discussed in [14,15].

Proposition 2 [5,6]. *Let* $f : [0, 1] \to [0, \infty]$ *be a conjunctive additive generator and let for all* $x, y \in [0, 1]$ *either* $f(x) + f(y) = f(z)$ *for some* $z \in [0, 1]$ *or* $f(x) + f(y) > f(z)$ *for all* $z \in]0, 1]$. *Then* $f \in \mathcal{F}$.

Note that continuity of a conjunctive additive generator f ensures the fulfilment of requirements of Proposition 2 and consequently ensures that f is an additive generator. Recall that by Ling [9] such f generates the continuous Archimedean t-norm. More, it can be shown that any $f \in \mathcal{F}$ fitting Proposition 2 yields an Archimedean t-norm. However, not all generated t-norms are Archimedean.

Example 1 [13]. Let $f : [0, 1] \to [0, \infty]$ be defined by

$$f(x) = \begin{cases} 3 - x & \text{if } x \in [0, 0.5[, \\ 1 - x & \text{if } x \in [0.5, 1] . \end{cases}$$

Then $f \in \mathcal{F}$ and $x = 0.5$ is the idempotent element of t-norm T_f , i.e., T_f is not Archimedean neither continuous.

3. Convergence of additive generators yielding the convergence of generated t-norms.

Let $f, f_1, f_2, \dots$ be additive generators and let $\lim_{n\to\infty} f_n(x) = f(x)$, for all $x \in [0, 1]$. Using similar arguments as in [11] we can shown that also $\lim_{n\to\infty} f_n^{(-1)}(x) = f^{(-1)}(x)$, for all $x \in [0, \infty]$, and consequently we obtain the next result.

Theorem 1. *Let* $f, f_1, f_2, \dots \in \mathcal{F}$ *be additive generators of t-norms* $T_f, T_{f_1}, T_{f_2}, \dots$, *respectively, and let* $\lim_{n\to\infty} f_n(x) = f(x)$, *for all* $x \in [0, 1]$. *Then* $\lim_{n\to\infty} T_{f_n}(x,y) = T_f(x,y)$, *for all* $(x,y) \in [0, 1]^2$.

It is easy to see that if $f \in \mathcal{F}$ and g differs from f only in the point 0, $g(0) > f(z)$ for all $z \in]0, 1]$, then also $g \in \mathcal{F}$ and $T_f = T_g$ .Therefore the convergence in Theorem 1 can be required for $x \in]0, 1]$ only, similarly as in [1,11]. Further, in [1,11] also the opposite claim with respect to Theorem 1 is proved in the case when $T, T_1, T_2, \ldots$ are continuous Archimedean t-norms, i.e., if $\lim_{n\to\infty} T_n = T$ then there are some additive generators $f, f_1, f_2, \ldots \in \mathcal{F}$ so that $T = T_f$, $T_n = T_{f_n}$, $n \in N$, and $\lim_{n\to\infty} f_n(x) = f(x)$, for all $x \in]0, 1]$. Whether this result can be extended for arbitrary generated t-norms is still an open problem.

Example 2. Define $f_n \in \mathcal{F}$, $n \in N$ by

$$f_n(x) = \begin{cases} 3 - x & \text{if } x \in [0, 0.5 - \frac{1}{n+2}[\,, \\ n + 3 - (2n+5)x & \text{if } x \in [0.5 - \frac{1}{n+2}, 0.5[\,, \\ 1 - x & \text{if } x \in [0.5, 1]\,. \end{cases}$$

Then f_n is a continuous additive generator for any $n \in N$ and T_{f_n} is a continuous Archimedean t-norm. Futher, $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in [0, 1]$, where $f \in \mathcal{F}$ is an additive generator introduced in Example 1. Consequently, $\lim_{n\to\infty} T_{f_n} = T_f$, i.e. the limit of continuous Archimedean t-norms $(T_n)_{n\in N}$ is a non-continuous and non-Archimedean generated t-norm T_f.

4. Concluding remarks.

We have shown that the pointwise convergence of additive generators to an additive generator results the pointwise convergence of the corresponding generated t-norms. Note that there are several additive generators generated the same generated t-norm, i.e., there is some freedom when choosing an additive generator to a given generated t-norm. As a direct convergence of Theorem 1 we see that if $f, f_1, f_2, \ldots, g, g_1, g_2, \ldots \in \mathcal{F}$, $T_{f_n} = T_{g_n}$, $n \in N$, and $\lim_{n\to\infty} f_n = f$, $\lim_{n\to\infty} g_n = g$, then also $T_f = T_g$.

Further note that the existence of $\lim_{n\to\infty} f_n$ itself is not enough to ensure that $\lim_{n\to\infty} T_{f_n}$ is a t-norm. Take, e.g.,

$$f_n(x) = \begin{cases} 1 - \frac{x}{2} & \text{if } x \in [0, 0.5 - \frac{1}{n+1}]\,, \\ \frac{n+7}{8} - \frac{n+3}{4}x & \text{if } x \in]0.5 - \frac{1}{n+1}, 0.5 + \frac{1}{n+1}[\,, \\ \frac{1-x}{2} & \text{if } x \in [0.5 + \frac{1}{n+1}, 1]\,. \end{cases}$$

Then f_n is a continuous additive generator for $n \in N$, and $\lim_{n\to\infty} f_n = f$, where

$$f(x) = \begin{cases} 1 - \frac{x}{2} & \text{if } x \in [0, 0.5[\,, \\ 0.5 & \text{if } x = 0.5\,, \\ \frac{1-x}{2} & \text{if } x \in]0.5, 1]\,. \end{cases}$$

However, $f \notin \mathcal{F}$ and $\lim_{n\to\infty} T_{f_n}$, though it exist, is not a t-norm. Anyway, f is a conjunctive additive generator and $\lim_{n\to\infty} T_{f_n} = T_f$, indicating that Theorem 1 can be extended to the class of all conjunctive additive generators and corresponding generated conjunctors.

Finally note that similar results can be expected also for generated aggregation operators [7,8].

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MILITARY ACADEMY, LIPTOVSKÝ MIKULÁŠ, SLOVAKIA
E-mail address: `rybarik@valm.sk`