A NOTE ON ARCHIMEDEAN TRIANGULAR NORMS

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ABSTRACT. The structure of continuous Archimedean t-norms is known for years [6,10]. However, the structure of Archimedean t-norms without continuity is not yet known. We will discuss the structure of left-continuous Archimedean t-norms and show that such t-norms are necessarily continuous, and hence generated by means of additive generators.

Key words. Triangular norm, Archimedean triangular norm, additive generator.

Triangular norms were introduced by Menger in 1942 [7] and in the present form by Schweizer and Sklar [9]. For the definition, basic notions and properties we refer the reader to the overview paper of Klement and Mesiar [2] and monographs [10,5].

It is well-known [6] that a t-norm is Archimedean and continuous if and only if it is generated by a continuous additive generator $f : [0,1] \to [0,\infty]$ that is strictly decreasing with $f(0) = 1$, in the following way :

$$T(x,y) = f^{(-1)}\left(f(x) + f(y)\right) , \tag{1}$$

where

$$f^{(-1)} : [0,\infty] \to [0,1], \quad f^{(-1)}(x) = \sup\{z \in [0,1] \; ; \; f(z) > x\}$$

is so-called pseudo-inverse of f, see [4].

The problem of the relationship of the Archimedean property of a t-norm T and its generatedness by means of (not necessarily continuous) additive generators is discussed, e.g., by Viceník in [14], compare also [3]. For the convenience of the reader we repeat two relevant results from [4,12,13].

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Proposition 1. Let $f : [0,1] \to [0,\infty]$ be a conjunctive generator, i.e., a strictly decreasing function with $f(1) = 0$ whose range is relatively closed under addition. Then the operator T defined by means of f by Eq. (1) is an Archimedean t-norm.

Recall that the range of f $Ran\, f$ is relatively closed under addition if for all $x,\ y \in [0,1]$, $f(x) + f(y) \in Ran\, f \cup [f(0^+), \infty]$.

Proposition 2. Let a left-continuous t-norm T be generated by an additive generator f. Then T is a continuous Archimedean t-norm.

Example 1. Let the function $f : [0,1] \to [0,\infty]$ be defined by

$$f(x) = \begin{cases} 1 - x & x \geq 0.5 \\ 3 - x & x < 0.5. \end{cases}$$

Then f generates via (1) a t-norm T with the property $T(0.5, 0.5) = 0.5$, see [12], which means that 0.5 is the idempotent element of T and thus T is not Archimedean.

Now, we are interested in a problem which is in some sense inverse to that one from the above proposition, namely, under which conditions an Archimedean t-norm is generated.

Theorem 1. Let T be a left-continuous Archimedean t-norm. Then T is a continuous generated t-norm.

Proof. Due to the Ling representation theorem [6] it is enough to prove the continuity of T, i.e., in our case the right-continuity of T.
Suppose that T is not right-continuous in some point (x, y). Because of the general properties of t-norms this point is from the open interval $]0,1[^2$ and

$$T(x,y) < T(x^+, y^+). \tag{2}$$

Now, take any strictly increasing sequence $\{z_n\}_{n=1}^{\infty} \subset]0,1[$, $z_n \nearrow 1$. Since T is Archimedean, for each $n \in \mathbb{N}$ there are uniquely determined constants $u_n, v_n \in \mathbb{N} \cup \{0\}$ such that

$$z_n^{(u_n+1)} \leq x < z_n^{(u_n)} \quad \text{and} \quad z_n^{(v_n+1)} \leq y < z_n^{(v_n)} \tag{3}$$

Recall that for $z \in [0,1]$, $z^{(0)} = 1$ and $z^{(n)} = T(z, z^{(n-1)})$ for $n \in \mathbb{N}$.
Due to (2) and (3), we obtain

$$z_n^{(u_n+v_n+2)} \leq T(x,y) < T(x^+, y^+) \leq z_n^{(u_n+v_n)}. \tag{4}$$

From the last part of (4), i.e., from the inequality

$$z_n^{(u_n+v_n)} \geq T(x^+, y^+),$$

we obtain

$$z_n^{(u_n+v_n+2)} \geq T\left(T(x^+,y^+), z_n^{(2)}\right).$$

Since $\lim_{n\to\infty} z_n = 1$ and T is a left-continuous t-norm, we have

$$\lim_{n\to\infty} T\left(T(x^+,y^+), z_n^{(2)}\right) = T(x^+,y^+)$$

and hence,

$$\liminf z_n^{(u_n+v_n+2)} \geq T(x^+,y^+). \tag{5}$$

Then, from (4) and (5) we have

$$\limsup z_n^{(u_n+v_n+2)} \leq T(x,y) < T(x^+,y^+) \leq \liminf z_n^{(u_n+v_n+2)}$$

for all $n \in \mathbb{N}$, which is a contradiction.

The right continuous t-norms need not be continuous as we can see for instance in the case of the drastic product T_D. Note that T_D is generated by any conjunctive additive generator f non-continuous in the point 1 and such that $f(0^+) \leq 2f(1^-)$, see [3,5]. In general, it is not known yet whether any Archimedean t-norm is generated. For example, the t-norm T^* introduced in [1], see also [11], defined for x, $y \in]0,1[$ by

$$T^*(x,y) = \sum_{n\in\mathbb{N}} \frac{1}{2^{(x_n+y_n)}},$$

where

$$x = \sum_{n\in\mathbb{N}} \frac{1}{2^{x_n}} \qquad \text{and} \qquad y = \sum_{n\in\mathbb{N}} \frac{1}{2^{y_n}}$$

are infinite dyadic expansions of x and y, respectively, is an Archimedean t-norm. It is neither right- nor left-continuous, and it is not known yet whether T^* is generated by means of some additive generator via (1).

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