GENERALIZED ORDINAL SUM THEOREM AND ITS CONSEQUENCE TO THE CONSTRUCTION OF TRIANGULAR NORMS

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Abstract

In this paper the well-known ordinal sum theorem of semigroups is generalized and applied to construct new families of triangular subnorms and triangular norms. Among them one can find several new families of left-continuous triangular norms too.

1 Introduction

Triangular norms (t-norms for short) have been introduced in the field of statistical (probabilistic) metric spaces in order to generalize the triangle inequality from metric spaces to statistical (probabilistic) metric spaces. For a nice overview on t-norms the reader is referred to [14]. Since then, they have been applied in several other fields of mathematics. Being fully ordered semigroups they are part of the classical algebra. A t-norm is defined on the unit square $[0,1]^2$ so it can be considered as solution of the associativity equation ([1, 9]). In fuzzy sets theory, together with their duals – the triangular conorms – they are extensively used to model the intersection and union of fuzzy subsets, respectively. In fuzzy logic (which is a many valued propositional logic with a continuum of truth values modelled by the unit interval) t-norms and t-conorms model the (semantic) interpretation of the logical conjunction and disjunction, respectively. In the field of decision making, fuzzy preference modeling uses t-norms and t-conorms as well ([7]). T-norms are applied in control, in the theory of non-additive measures and integrals ([17]) and so on.

The aim of this paper is to generalize the well-known ordinal sum theorem of semigroups and to apply it for constructing new triangular subnorms and triangular norms. Perhaps the most important consequence is that a new construction method for *left*-continuous triangular norms arise.

We remark that other ordinal sum-type theorems can be introduced as well. For instance, ordinal sums of a t-norm and a t-conorm (see [2]) led to the conjunctive and disjunctive uninorms which had been discussed in [8].

2 Preliminaries

A *t-norm* is a function $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms (T1)-(T4) are satisfied:

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(*T1*)	*Symmetry*	$T(x,y) = T(y,x)$
(*T2*)	*Associativity*	$T(x,T(y,z)) = T(T(x,y),z)$
(*T3*)	*Monotonicity*	$T(x,y) \leq T(x,z)$ whenever $y \leq z$
(*T4*)	*Boundary condition*	$T(x,1) = x$
(*T4'*)	*Boundary condition*	$T(x,0) = 0$
(*T4''*)	*Range condition*	$T(x,y) \leq \min(x,y)$.

It is immediate to see that (T3) and (T4) imply (T4'), and that (T1), (T3) and (T4) imply (T4").

Triangular subnorms were introduced in [12] where they have been used in the so-called rotation-annihilation construction in order to construct new families of left-continuous t-norms T satisfying $T(x,y) = 0$ if and only if $x + y \leq 1$: A *triangular subnorm* (t-subnorm for short) is a function $T : [0,1]^2 \to [0,1]$ such that for all $x,y,z \in [0,1]$ axioms (T1), (T2), (T3) and (T4") are satisfied.

First, we recall a result from [4] which generalizes a result of [3] concerning the ordinal sum of two disjoint semigroups (see as well [5]). This theorem discusses a certain way of constructing a new semigroup from a family of semigroups.

Theorem 1 *Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha\beta}\}$, where $x_{\alpha\beta}$ is both the unit element of G_α and the annihilator of G_β, and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha\beta}\}$. Put $X = \bigcup_{\alpha\in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \textit{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \textit{if } (x,y) \in X_\alpha \times X_\beta \textit{ and } \alpha < \beta, \\ y & \textit{if } (x,y) \in X_\alpha \times X_\beta \textit{ and } \alpha > \beta. \end{cases} \quad (1)$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

Ordinal sum theorems have been adapted into the field of t-norms by many authors (first in [18] for two – hence by induction, for a finite number of – semigroups; and in the below-described form first in [9]) and has been spreaded widely in the literature with the following formulation under the name *ordinal sum theorem for t-norms.*

Theorem 2 *Suppose that $\{[a_i, b_i]\}_{i\in K}$ is a countable family of non-overlapping, closed, proper subintervals of $[0,1]$, denoted by $\mathcal{I}$. With each $[a_i, b_i] \in \mathcal{I}$ associate a t-norm T_i. Let T be a function defined on $[0,1]^2$ by*

$$T(x,y) = \begin{cases} a_m + (b_m - a_m)T_m\left(\frac{x-a_m}{b_m-a_m}, \frac{y-a_m}{b_m-a_m}\right) & \textit{if } (x,y) \in [a_m,b_m]^2 \\ \min(x,y) & \textit{otherwise} \end{cases} . \quad (2)$$

Then T is a t-norm and called the ordinal sum *of $\{([a_i,b_i],T_i)\}_{i\in K}$ and each T_i is called a* summand.

3 The generalization

A closer look at Theorem 1 reveals immediately that Theorem 2 is not the most general translation of it. A more general adaptation reads as follows:

Suppose that $\{[a_i, b_i]\}_{i \in K}$ *is a countable family of non-overlapping, closed, proper subintervals of* $[0,1]$, *denoted by* $\mathcal{I}$. *With each* $[a_i, b_i] \in \mathcal{I}$ *associate a t-subnorm* T_i *where for each* $[a_i, b_i], [a_j, b_j] \in \mathcal{I}$ *with* $b_i = a_j$ *we have that* T_i *is a t-norm and where for* $[a_i, 1] \in \mathcal{I}$ *we have that* T_i *is a t-norm. Then* $T : [0,1]^2 \to [0,1]$ *given by (2) is a t-norm.*

The reason of this fact might be that in the early investigations of t-norms their continuity played a central role. It is well-known from [16] (her result is based essentially on [15]) that any continuous t-norm can be represented as ordinal sum of continuous Archimedean t-norms. Continuous t-norms have become well-understood by this theorem and hence they have been widely used in mathematical and practical applications. Nevertheless, in many applications the condition of continuity is not necessary, only the left-continuity of the t-norm is required. But for a long time there hasn't been known any example of left-continuous t-norms which are not continuous. The first results in this direction are in [6, 11, 12]. Recently, many mathematical fields – such as probabilistic metric spaces, fuzzy logics, fuzzy preference modeling, the theory of measure-free conditioning – are calling for new *left-continuous* t-norms.

In what follows, we generalize Theorem 1, then we apply it to t-norms. As a consequence, an infinite number of new t-norms are given birth.

Theorem 3 (Generalized Ordinal Sum Theorem) *Let* $A \neq \emptyset$ *be a totally ordered set and* $(G_\alpha)_{\alpha \in A}$ *with* $G_\alpha = (X_\alpha, *_\alpha)$ *be a family of semigroups. Assume that for all* $\alpha, \beta \in A$ *with* $\alpha < \beta$ *the sets* X_α *and* X_β *are either disjoint or that* $X_\alpha \cap X_\beta = \{x_{\alpha\beta}\}$, *where* $x_{\alpha\beta}$ *is the annihilator of* G_β, *and* $x_{\alpha\beta}$ *is the unit element of* G_α *when* G_β *has zero divisors (that is, if there exist* $x, y \in G_\beta$ *with* $x \neq x_{\alpha\beta}$, $y \neq x_{\alpha\beta}$ *such that* $x *_\beta y = x_{\alpha\beta}$*), and where for each* $\gamma \in A$ *with* $\alpha < \gamma < \beta$ *we have* $X_\gamma = \{x_{\alpha\beta}\}$. *Put* $X = \bigcup_{\alpha \in A} X_\alpha$ *and define the binary operation* $*$ *on* X *by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha < \beta \text{ and } x \neq x_{\alpha\beta},\ y \neq x_{\alpha\beta} \text{ for some } \alpha, \beta \in A, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha > \beta \text{ and } x \neq x_{\alpha\beta},\ y \neq x_{\alpha\beta} \text{ for some } \alpha, \beta \in A. \end{cases} \quad (3)$$

Then $G = (X, *)$ *is a semigroup. The semigroup* G *is commutative if and only if for each* $\alpha \in A$ *the semigroup* G_α *is commutative.*

Proof. For all indeces $\alpha, \beta \in A$ with $\alpha < \beta$, $X_\alpha \cap X_\beta = \{x_{\alpha\beta}\}$ for which X_β has no zero divisors, replace the semigroups $G_\beta = (X_\beta, *_\beta)$ by the semigroup $(X_\beta \setminus \{x_{\alpha\beta}\}, *_\beta|_{X_\beta \setminus \{x_{\alpha\beta}\}})$. It is easy to verify that Theorem 1 can be applied to the obtained family of semigroups which thus concludes the proof. ∎

Corollary 1 (Ordinal Sum Theorem for t-subnorms) *Suppose that* $\{[a_i, b_i]\}_{i \in K}$ *is a countable family of non-overlapping, closed, proper subintervals of* $[0,1]$, *denoted by* $\mathcal{I}$. *With each* $[a_i, b_i] \in \mathcal{I}$ *associate a t-subnorm* T_i *where for each* $[a_i, b_i], [a_j, b_j] \in \mathcal{I}$ *with* $b_i = a_j$ *and with zero divisors in* T_j *we have that* T_i *is a t-norm. Let* T *be a function defined on* $[0,1]^2$ *by*

$$T(x,y) = \begin{cases} a_m + (b_m - a_m) T_m \left(\frac{x - a_m}{b_m - a_m}, \frac{y - a_m}{b_m - a_m} \right) & \text{if } (x,y) \in]a_m, b_m]^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \quad (4)$$

Then T is a t-subnorm and called the ordinal sum *of* $\{([a_i, b_i], T_i)\}_{i\in K}$ *and each* T_i *is called a* summand.

Corollary 2 (Generalized Ordinal Sum Theorem for t-norms) *Suppose that* $\{[a_i, b_i]\}_{i\in K}$ *is a countable family of non-overlapping, closed, proper subintervals of* $[0, 1]$*, denoted by* $\mathcal{I}$*. With each* $[a_i, b_i] \in \mathcal{I}$ *associate a t-subnorm* T_i *where for each* $[a_i, b_i], [a_j, b_j] \in \mathcal{I}$ *with* $b_i = a_j$ *and with zero divisors in* T_j *we have that* T_i *is a t-norm and for* $[a_i, 1] \in \mathcal{I}$ *we have that* T_i *is a t-norm. Let* T *be a function defined on* $[0, 1]^2$ *by (4). Then* T *is a t-norm.*

Proof. Only the boundary condition of t-norms has to be verified which follows immediately from the last condition. ■

Summarizing, we can use t-subnorms as summands instead of t-norms. The only restrictions are: The last summand (if it exists) has to be a t-norm when we want to construct a t-norm and a summand always has to be a t-norm if there is another summand with zero divisors just above it.

Note that the generalized ordinal sum construction preserves left-continuity. The proof of the next proposition is obvious.

Proposition 1 *An ordinal sum (given by Corollary 1 or Corollary 2) is left-continuous if and only if all of its summands are left-continuous.*

By using Theorem 2 we can construct several new t-norms. For instance, the drastic product t-norm given by

$$T_{\mathbf{D}}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1[^2 \\ \min(x, y) & \text{otherwise} \end{cases}$$

can be replaced by the trivial t-subnorm $T_{\mathbf{d}}(x, y) \equiv 0$ and the t-norm $\{([0, \lambda], T_{\mathbf{D}}), ([\lambda, 1], \min)\}$ (which was proposed in [18]) can be modified to $\{([0, \lambda], T_{\mathbf{d}}), ([\lambda, 1], \min)\}$ and one obtains a left-continuous t-norm.

Non-trivial examples of t-subnorms are e.g. (ε is a fixed real from $[0, 1[$)

$$T(x, y) = \varepsilon x y,$$

$$T(x, y) = \max(0, x + y - 1 - \varepsilon).$$

We draw the attention of the reader to the fact, that the construction in [10] (Theorem 2) (see as well [14] and [13] for an exhaustive overview and development) produces t-subnorms as well if the boundary of the resulted t-norm is not redefined (in the formula which can be found in the cited references). Moreover, if one starts with a left-continuous t-(sub)norm, then the just mentioned construction (again without the separate definition on the boundary) produces a left-continuous t-subnorm.

4 Conclusion

The classical ordinal sum theorem is generalized. As a consequence, it is pointed out that t-subnorms can be used (with a little restriction) instead of t-norms as summands in the ordinal sum theorem of t-norms. Several new families of t-norms (among them *left*-continuous ones) can be introduced by this observation.

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