$\mathcal{T}$-FUZZY PARTITIONS AND $\mathcal{T}$-FUZZY EQUIVALENCE RELATIONS

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1. Introduction

Partitions and equivalence relations belong to the basic concepts of the standard mathematics based on Cantorian sets. Recall only that for a given nonempty universe $\mathbf{X}$, a subset $\mathbf{P} \subset 2^{\mathbf{X}}$ is called a partition if it is a disjoint covering of $\mathbf{X}$ not containing the empty set, i.e. $P \cap Q = \emptyset$ for all $P \neq Q$, $P, Q \in \mathbf{P}$, $\bigcup_{P \in \mathcal{P}} P = \mathbf{X}$ and $P \neq \emptyset$ for all $P \in \mathbf{P}$. Moreover, a subset $\mathbf{E} \subseteq \mathbf{X}^2$ is called an equivalence relation if $(x,x) \in \mathbf{E}$ for all $x \in \mathbf{X}$ (reflexivity), $(x,y) \in \mathbf{E}$ implies $(y,x) \in \mathbf{E}$ (symmetry), and (x,y) and $(y,z) \in \mathbf{E}$ imply that also $(x,z) \in \mathbf{E}$ (transitivity). Recall that there is a one-to-one correspondence between partitions and equivalence relations. Namely, for $\mathbf{P}$ a partition and $\mathbf{E}$ an equivalence relation (on $\mathbf{X}$), the system

$$\mathbf{P}_{\mathbf{E}} = \{\mathbf{E}_x;\ x \in \mathbf{X}\}\ ,\ \mathbf{E}_x = \{y;\ (x,y) \in \mathbf{E}\},$$

is a partition, the subset

$$\mathbf{E}_{\mathbf{P}} = \{(x,y);\ x,\ y \in P \text{ for some } P \in \mathbf{P}\} \subseteq \mathbf{X}^2$$

is an equivalence relation, and

$$\mathbf{P}_{\mathbf{E}_{\mathbf{P}}} = \mathbf{P}\ ,\ \mathbf{E}_{\mathbf{P}_{\mathbf{E}}} = \mathbf{E}\ .$$

Fuzzy partitions were introduced for the first time in 1969 by Ruspini [7]. However, Ruspini's approach has no counterpart in equivalence relations domain (Ruspini's partitions are systems and not sets in general, they can contain two equal members in general).

Fuzzy equivalence relations were introduced for the first time by Zadeh [9] in 1971. The transitivity in Zadeh's approach is based on the strongest t-norm T_M. Though Zadeh has not developed the partition counterpart of his approach, it can be defined as we will see later. Many other approaches to the fuzzy partitions and fuzzy equivalence relations have been introduced so far, see e.g. [1,3,4,5]. For an overview of several types of fuzzy partitions we recommend [2].

The aim of this note is to discuss the fuzzification of the concepts of partitions and equivalence relations preserving their classical relationship. Our approach is based on an operator $\tau: [0,1]^2 \longrightarrow [0,1]$ which can be understood as some kind of fuzzy conjunction. More, we want to stress a sound approach to fuzzification of any standard notion (which is often not fulfilled in several domains of applications of the fuzzy sets theory).

2. τ-fuzzy partitions and τ-fuzzy equivalence relations

In what follows, we adopt an approach to defining fuzzy partitions and fuzzy equivalence relations based on the ideas of Thiele and Schmechel [8].

Definition 1. Let $\tau: [0,1]^2 \longrightarrow [0,1]$ be a given binary operation. A fuzzy subset $\mathcal{E}$ of $\mathbf{X}^2$, $\mathcal{E}: \mathbf{X}^2 \longrightarrow [0,1]$, will be called a τ-fuzzy equivalence relations if it is reflexive, i.e., $\mathcal{E}(x,x) = 1$ for all $x \in \mathbf{X}$, symmetric, i.e., $\mathcal{E}(x,y) = \mathcal{E}(y,x)$ for all $x,\ y \in \mathbf{X}$, and τ-transitive, i.e., for all $x,\ y,\ z \in \mathbf{X}$ it is

$$\tau(\mathcal{E}(x,y),\mathcal{E}(y,z)) \le \mathcal{E}(x,z) \quad . \qquad (1)$$

■

Recall that the τ-transitivity (1) means that if x and y are in relation $\mathcal{E}$ in degree $a = \mathcal{E}(x,y)$ **AND** y and z are in relation $\mathcal{E}$ in degree $b = \mathcal{E}(y,z)$ then x and z should be in relation $\mathcal{E}$ at least in degree $c = \tau(a,b)$, i.e., $\mathcal{E}(x,z) \ge c$. In the next we will look for the reasonable choice of the operator τ.

Definition 2. Let $\tau: [0,1]^2 \longrightarrow [0,1]$ be a given binary operation. A subset $\mathcal{P} \subset \mathcal{F}(\mathbf{X})$ (i.e., a set which elements are fuzzy subsets of $\mathbf{X}$) will be called a τ-fuzzy partition if each of its elements has non-empty kernel, i.e., for all $P \in \mathcal{P}$ there is $x \in \mathbf{X}$ so that $P(x) = 1$, it is a covering of $\mathbf{X}$, i.e., for any $x \in \mathbf{X}$ there is some $P \in \mathcal{P}$ so that $P(x) = 1$, and it fulfills the following τ-disjointedness property

if $P(x) = 1$ for some $P \in \mathcal{P}$, $x \in \mathbf{X}$, then for any $Q \in \mathcal{P}$ and $y \in \mathbf{X}$ it is

$$\tau(Q(x),Q(y)) \le P(y) \quad . \qquad (2)$$

■

Note that the property (2) for an appropriate choice of τ really extends the mutual disjointedness of members of standard partitions, see [8] for deeper discussion. A reasonable choice of an operator τ will be discussed latter.

3. Sound fuzzification

For any standard mathematical concept we find indispensable the following three requirements to get a sound fuzzification:

i) each value $a \in [0,1]$ is acceptable in the range of the fuzzy concept;

ii) each element of the standard concept is also an element of the fuzzy concept;

iii) each crisp element of the fuzzy concept is also an element of the standard concept.

In the case of τ-fuzzy partitions and τ-fuzzy equivalence relations, this means that for any $a \in [0,1]$ there is some $\mathcal{P}$ and $\mathcal{E}$ so that for some $P \in \mathcal{P}$ and for some $x \in \mathbf{X}$ it is $P(x) = a$, and for some y, $z \in \mathbf{X}$ it is $\mathcal{E}(y,z) = a$. Further, each standard partition (equivalence relation) should be also a τ-fuzzy partition (a τ-fuzzy equivalence relation). Finally, each crisp τ-fuzzy partition (crisp τ-fuzzy equivalence relation) should be also a standard partition (a standard equivalence relation).
More, in our specific case, we will require also the next properties:

iv) our fuzzification is fitting, i.e., for any τ-fuzzy partition $\mathcal{P}$, if for some $x \in \mathbf{X}$ and $P, Q \in \mathcal{P}$ it is $P(x) = Q(x) = 1$ the $P = Q$; for any τ-fuzzy equivalence relation $\mathcal{E}$, if $\mathcal{E}(x,y) = 1$ then $\mathcal{E}_x = \mathcal{E}_y$, where $\mathcal{E}_x$ is a fuzzy subset of $\mathbf{X}$ defined by $\mathcal{E}_x(t) = \mathcal{E}(x,t)$, $t \in \mathbf{X}$;

v) our fuzzification is duality fitting, i.e., there is a one-to-one correspondence between τ-fuzzy partitions and τ-fuzzy equivalence relations given by $\mathcal{P}_{\mathcal{E}} = \{\mathcal{E}_x;\ x \in \mathbf{X}\}$ and $\mathcal{E}_{\mathcal{P}}(x,y) = P(x,y)$ where P is a member of $\mathcal{P}$ for which $P(x) = 1$.

In the next section, we will give the weakest requirements on τ ensuring the validity of some of the above mentioned properties.

4. Appropriate choice of τ

Note that first the last property v) puts together fuzzy partitions and fuzzy equivalence relations. Therefore, for ensuring some of the properties i)-iv), the requirement on τ in the case of partitions may differ from those in the case of equivalence relations. Namely, equivalence relations are a priori symmetric which should be reflected by τ and what is not the case of partitions.

Proposition 1.

eq) Let $\tau(1,a) \le a$ and $\tau(a,1) \le a$ for all $a \in [0,1]$. Then the concept of τ-fuzzy equivalence relations fits the property i).

pa) Let for all $a \in [0,1]$, $\tau(1,a) \le a$ and there is some $b \in [0,1]$ such that $\tau(a,1) \le b$ and $\tau(b,1) \le a$. Then the concept of τ-fuzzy partitions fits the property i).■

Note that if card $\mathbf{X} = 2$ then $\mathcal{E}$ can be described by a matrix 2x2 and under requirements of Proposition 1, we have $\mathcal{E} = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ for some $a \in [0,1]$. It can be shown that these τ-equivalence relations fit also requirements ii) and iii).

Similarly, each $\mathcal{P} \neq \mathbf{X}$ is given by $\mathcal{P} = \{P,Q\}$ where $P = (1,a)$ and $Q = (b,1)$ for some $a, b \in [0,1]$ such that $\tau(a,1) \le b$ and $\tau(b,1) \le a$. However, then also $\tau \equiv 0$ fits pa) in Proposition 1, admitting any $a, b \in [0,1]$. Consequently, there is a crisp τ-fuzzy partition $\mathcal{P} = \{P,Q\}$, $P = (1,1)$, $Q = (1,0)$, which is not a standard partition of $\mathbf{X}$.

Proposition 2.

eq) Let $\tau(1,a) \le a$ and $\tau(a,1) \le a$ for all $a \in [0,1]$. Then the concept of τ-fuzzy equivalence relations fits the properties i), ii) and iii).

pa) Let for all $a \in [0,1]$, $\tau(1,a) \le a$ and there is some $b \in [0,1]$ such that $\tau(a,1) \le b$ and $\tau(b,1) \le a$. More, let $\tau(0,0) = 0$ and let $\tau(1,1) > 0$. Then the concept of τ-fuzzy partitions fits the properties i), ii) and iii). ■

For the "fitting" fuzzy extensions of equivalence relations and partitions we have the next weakest requirements on τ.

Proposition 3.

eq) Let $\tau(1,a) = a$ and $\tau(a,1) \le a$ for all $a \in [0,1]$ (or vice versa) and let $\tau(0,0) = 0$. Then the concept of τ-fuzzy equivalence relations fits the properties i)-iv).

pa) Let for all $a \in [0,1]$, $\tau(1,a) = a$ and there is some $b \in [0,1]$ such that $\tau(a,1) \le b$ and $\tau(b,1) \le a$. More, let $\tau(0,0) = 0$. Then the concept of τ-fuzzy partitions fits the properties i)-iv). ■

Till now the requirements on τ in the case of partitions differs from those in the case of equivalence relations. As an example of τ fitting the requirements in Proposition 4 (both in eq) and pa) case) recall the operator

$\tau^*(a,b) = \begin{cases} 0 & \text{if } a < 1 \\ b & \text{if } a = 1 \end{cases}$, which can be obtained from the drastic product if we apply the residuation to get an implicator and backwards residuation to get a conjunctor. However, τ^*-concept does not admit the one-to-one correspondence between τ^*-fuzzy partitions and τ^*-fuzzy equivalence relations. This is ensured only if τ fulfills the following requirements.

Proposition 5. Let 1 be the neutral element of τ, i.e., $\tau(1,a) = \tau(a,1) = a$ for all $a \in [0,1]$, and let $\tau(0,0) = 0$. Then the concepts of τ-fuzzy partitions and τ-fuzzy equivalence relations are in a one-to-one correspondence and they fit all requirements i)-v). ■

5. Cocluding remarks

It is immediate that the weakest operator τ leading to a sound fuzzy concept of fuzzy partitions and fuzzy equivalence relations (i.e., fulfilling requirements in Proposition 5) is the drastic product $\mathbf{T}_D$, i.e., the weakest t-norm, while the strongest is the operator τ_S given by

$$\tau_S(a,b) = \begin{cases} \min(a,b) & \text{if } \max(a,b) = 1 \\ 0 & \text{if } a = b = 0 \\ 1 & \text{otherwise} \end{cases} .$$

More, an operator τ fulfills the requirements of Proposition 5 if and only if $\mathbf{T}_D \le \tau \le \tau_S$. Note that the operator τ_S is not monotone. If we additionally require also the monotonicity of τ, then such strongest operator is the strongest t-norm $\mathbf{T}_M$ and the inequalities $\mathbf{T}_D \le \tau \le \mathbf{T}_M$ give the sufficient and necessary condition for a monotone operator τ to fit Proposition 5. Therefore the concept of T-fuzzy partitions and T-fuzzy equivalence relations with T a t-norm proposed by several authors is sound (see e.g.

[1,3,4,5,8,9])! However, we can take as τ any (idempotent) aggregation of several t-norms, for example such as min T_i, max T_i, arithmetic mean of t-norms, geometric mean of t-norms, etc. Similarly, we can take as τ any generated conjunctor. Several examples, characterization of τ-fuzzy partitions and τ-fuzzy equivalence relations will be the topic of our forthcoming paper [6].

Acknowledgment. The work on this paper was partially supported by grants VEGA 2/6087/99, 1/4064/97 and GAČR 402/99/0032.

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