A NOTE ON THE RELEVANCY TRANSFORMATION OPERATOR IN FUZZY REASONING

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ABSTRACT. To obtain the effective rule output from the rule relevancy and the rule consequent in fuzzy modelling inference process we can use an operator which is called RET operator. The possibilities of its construction are studied.

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Fuzzy modelling rule based inference process (FUMIP) can considered in four-step algorithm (Yager [12]), (Yager and Filev [13]). Suppose that a rule base consists of a collection of n rules in the form

$$\text{if } V \text{ is } A_i \text{ then } U \text{ is } B_i,$$

where A_i and B_i are fuzzy sets of X and Y respectively.

1.step : (matching step) Determine a relevance τ_i of each rule for a given input value.
E.g. $\tau_i = A_i(x^*)$, where x^* is a crisp input or $\tau_i = \mathrm{Max}\,(A_i \wedge C)$, where C is an fuzzy input [5].

2.step : Determine an effective fuzzy output of each rule. We assume that this process is pointwise, i.e.

$$F_i(y) = h(\tau_i\,, B_i(y))$$

E.g. $F_i(y) = \mathrm{Min}\{\tau_i, B_i(y)\}$ or $F_i(y) = \mathrm{Max}\{(1-\tau_i), B_i(y)\}$ (Zadeh [14])

3. step: Aggregation of individual rule outputs

$$F(y) = \mathrm{Agg}(F_1(y), F_2(y), ..., F_n(y))$$

E.g.

$$F(y) = \mathrm{Min}\{F_1(y), F_2(y), ..., F_n(y)\}$$

or

$$F(y) = \mathrm{Max}\{F_1(y), F_2(y), ..., F_n(y)\}$$

or

$$F(y) = \mathrm{Uni}\,(F_1(y), F_2(y), ..., F_n(y))$$

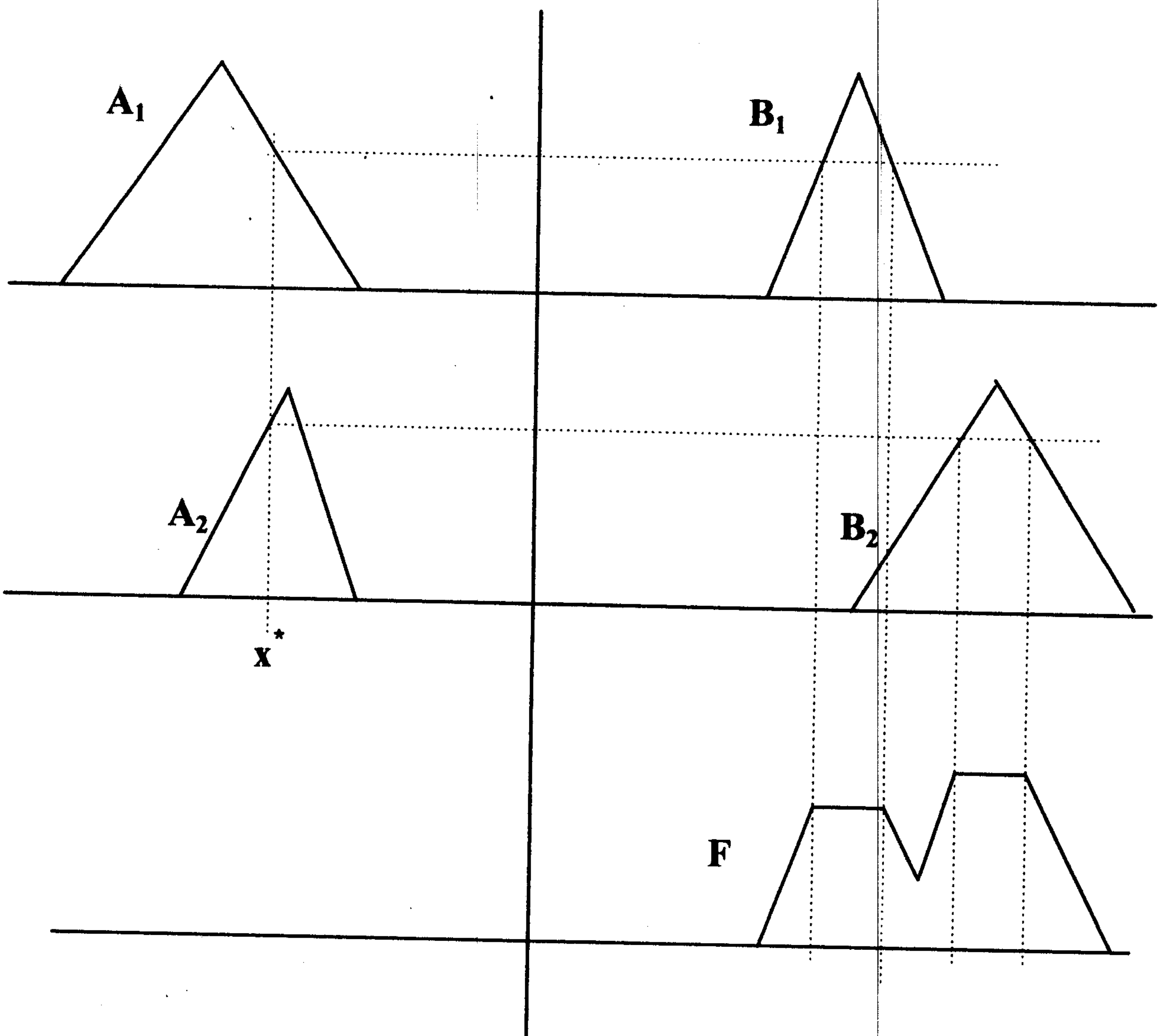
where the n-ary operation Uni is an uninorm [2], [3], [12].

4. step: Defuzzification

Example 1. The next picture shows a simple example of FUMIP with two fuzzy rules in the form: If V is A_i then U is B_i (A_i, B_i are triangular fuzzy numbers [6]), with a crisp input x^*, matching values $\tau_i = A_i(x^*)$, effective fuzzy inputs $F_i(y) = \mathrm{Min}\{\tau_i, B_i(y)\}$ and with an aggregation operator

$$F(y) = \mathrm{Max}\{F_1(y), F_2(y)\}$$

The defuzzification step is omitted.



Picture 1

We focus our interest to the second and third steps and to the relations between them. Assume that there exists some rule with zero relevancy, i.e. $\tau_n = 0$. If $\tau_n = 0$ then n-th rule provides no information regarding output values, i.e., F_n should not make any distinction between the values. Therefore

$$F_n(y) = \text{const} = c$$

It is a natural requirement that such output plays no role in the aggregation, i.e.

$$Agg(F_1(y), F_2(y), \ldots, F_{n-1}(y), c) = Agg(F_1(y), F_2(y), \ldots, F_{n-1}(y))$$

It means that c would be the identity element of the aggregation [12], [7].

Example 2. Simple examples of aggregation operators can be create from associative binary operations:

a) Agg $(x_1,x_2,...,x_n) = T(x_1,x_2,...,x_n)$, where T is a t-norm (associative commutative binary operation on the unit square for which $T(x,1) = x$), $T(x_1,x_2,...,x_n) = T(T(x_1,x_2,...,x_{n-1}),x_n)$, $T(x) = x$. In this case the identity element c = 1.

b) Agg $(x_1,x_2,...,x_n) = S(x_1,x_2,...,x_n)$, where S is a t-conorm (associative commutative binary operation on the unit square for which $S(x,0) = x$), $S(x_1,x_2,...,x_n) = S(S(x_1,x_2,...,x_{n-1}),x_n)$, $S(x) = x$. In this case the identity element c = 0.

c) Agg $(x_1,x_2,...,x_n) = Uni(x_1,x_2,...,x_n)$, where Uni is a uninorm, i.e. associative commutative binary operation on the unit square for which $Uni(x,e) = x$ for a given element $e \in (0,1)$ and any $x \in [0,1]$, $Uni(x_1,x_2,...,x_n) = Uni(Uni(x_1,x_2,...,x_{n-1}),x_n)$, $Uni(x) = x$. In this case the identity element c = e.

d) (three π operator) [12], [3]

$$P(x_1,x_2,...,x_n) = \frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i + \prod_{i=1}^{n}(1-x_i)} \quad \text{if} \quad \{0;1\} \not\subset \{x_1,x_2,...,x_n\} \quad \text{else}$$

$P(x_1,x_2,...,x_n) = a$, where $a \in \{0; 1\}$. This aggregation operator has the identity element c = 0,5.

The aggregation operators in Example 2 are associative. The next example shows a non associative aggregation operator [9].

Example 3. The aggregation operator $L(x_1,x_2,...,x_n) = \mathrm{Max}\left\{0, \mathrm{Min}\left\{1, \sum_{i=1}^{n} x_i - \frac{n-1}{2}\right\}\right\}$ has the identity element c = 0.5. See also [8] for another interesting examples of non associative aggregation operators.

Definition of RET operator

To full determination of this type of FUMIP we need both an aggregation operator Agg (third step) and a binary operator h (second step) which have the same special element c. Now we shall concentrate our attention on the operator h.

Generally, h is a binary operation on the interval [0,1] having several specific properties [12].

Definition 1. The operator $h: [0;1]^2 \to [0;1]$ is called Relevancy Transformation operator (RET) with respect to the given identity element $c \in [0;1]$ iff

1) $h(1,b) = b$
2) $h(0,b) = c$
3) If $b_1 < b_2$ then $h(a,b_1) \le h(a,b_2)$
4) $b \ge c$ then $a_1 < a_2 \Rightarrow h(a_1,b) \le h(a_2,b)$
5) $b \le c$ then $a_1 < a_2 \Rightarrow h(a_1,b) \ge h(a_2,b)$

for any $a,b,a_1,a_2,b_1,b_2 \in [0;1]$.

The first requirement says: if the i-th rule is completely fired ($\tau_i = 1$) then the effective input is B_i. The second one was already explained. Third condition is a reflection of the requirement that we should not inverse the preference ordering of output values. The last conditions are called consistency in the antecedent argument. Remark that 4) and 5) imply h(a,c) = c for any a $\in [0,1]$. The next example gives some trivial models of RET operator

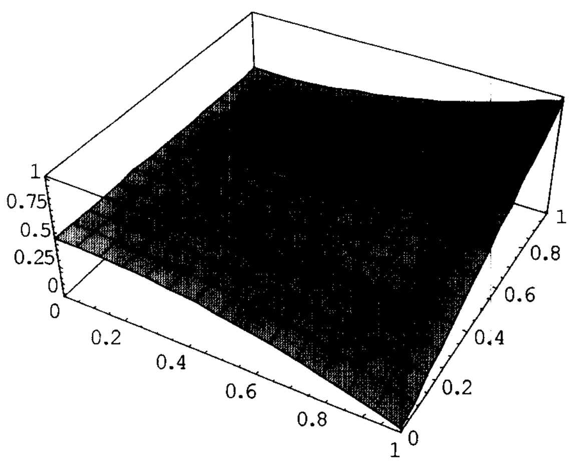
Example 4. ([12])
a) Let T be a t-norm. Then h(a,b) = T(a,b) is RET operator with c = 0
b) Let S be a t-conorm. Then h(a,b) = S(1-a,b) is RET operator with c = 1
c) $h(a,b) = \text{Max}\{\text{Min}\{a,b\}, \text{Min}\{1-a,c\}, \text{Min}\{b,c\}\}$ is RET operator for any $c \in [0,1]$
d) h(a,b) = ab + (1-a)c is RET operator for any $c \in [0,1]$ (see Picture 4)

One class of RET operators

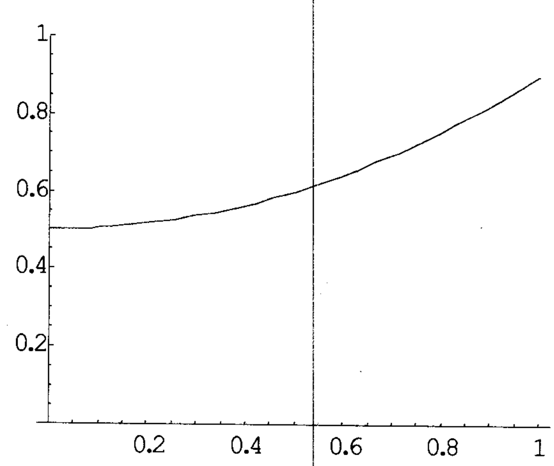
There exist many binary operations on the unit square having required properties. Now we give some class of polynomial binary operations which are RET operators

$$h(a,b) = c + (b - c)\, a^t = a^t b + (1 - a^t)\, c$$

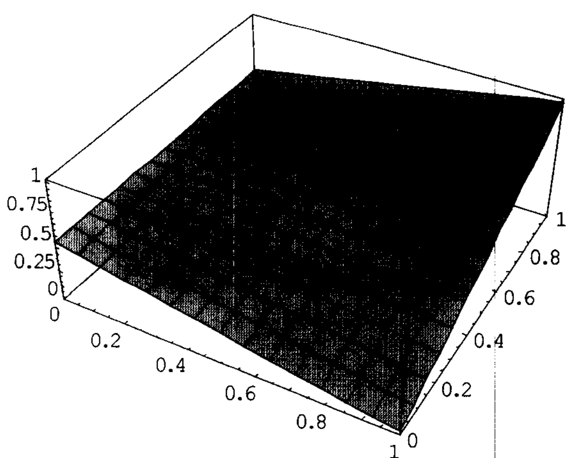
where c is given identity element, $c \in [0, 1]$, $t \in (0, \infty)$. The next pictures (Pictures 2 - 5) show this operator for c = 0.5 , t = 2 and t = 1 and their section for b = 0.9



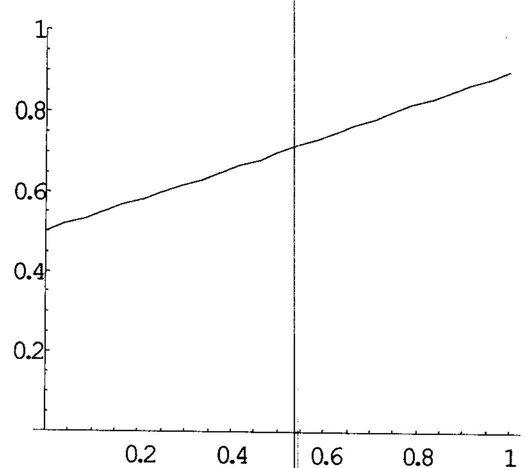
Picture 2. Graph h(a,b), c = 0.5, t = 2



Picture 3. Graph h(a, 0.9), c = 0.5, t = 2



Picture 4. Graph h(a,b), c = 0.5, t = 1



Picture 5. Graph h(a, 0.9), c = 0.5, t = 1

It is possible to generalise this class of RET operators by adding a lower limit β of acceptance of a rule and an upper limit α of full acceptance of a rule. It means, firing values less then β are vanishing and firing values greater then α are considered to be full.

$$h(a,b) = \begin{cases} c & a \in [0,\beta] \\ c + \dfrac{b-c}{\alpha^t - \beta^t}(a^t - \beta^t) & a \in [\beta,\alpha] \\ b & a \in [\alpha,1] \end{cases}$$

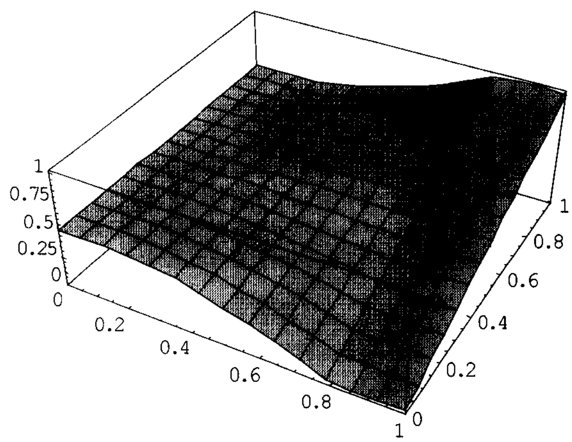
where c - identity element ; $c \in [0, 1]$.

β - lower limit of acceptance of a rule

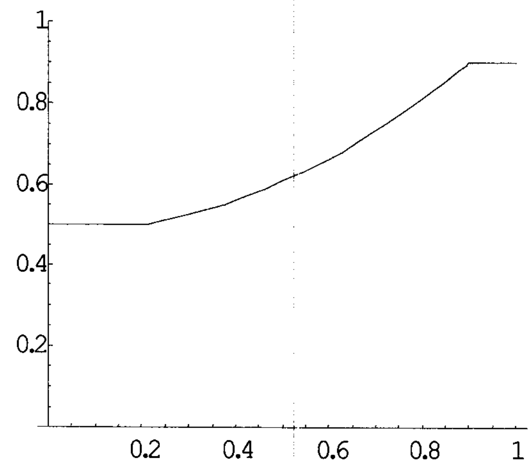
α - lower limit of full acceptance of a rule, $\beta < \alpha$

$t \in (0, \infty)$.

The next pictures (Pictures 6-7) show some special cases of such RET operators



Picture 6. Graph h(a,b)
$c = 0.5$, $\beta = 0.2$, $\alpha = 0.8$, $t = 2$



Picture 7. Graph h(a, 0.9)
$c = 0.5$, $\beta = 0.2$, $\alpha = 0.9$, $t = 2$

RET operators generated from t-norms

We have already seen that any t-norm can be considered as a RET operator with c = 0. Motivated by Example 4 we can tray to construct RET operator h as follows:

$$h(a,b) = T(a,b) + T(1-a,c)$$

where T is some t-norm, c is a given element of the unit interval . It can be easily shown that for such operator h 1), 2), 3) hold however 4) and 5) may be violated. It would be interesting to characterise t-norm for which the operator h(a,b) = T(a,b) + T(1-a,c) is a RET operator. The next Lemma is a direct consequence of the fact that RET operator satisfies h(a,c) = c.

Lemma. Let h(a,b) = T(a,b) + T(1-a,c) be a RET operator for given $c \in [0, 1]$, T be a t-norm. Then for any $a \in [0, 1]$ it holds T(a,c) + T(1-a,c) = c.

Theorem. Let T be a t-norm with a continuous convex generator such that T(a,c) + T(1-a,c) = c for some $c \in (0; 1)$ and all $a \in [0;1]$. Then h(a,b) = T(a,b) + T(1-a, c) is a RET operator with respect to the element c.

Proof. Consider a t-norm T with continuous convex generator. Then T is a copula ([10], Chapter 6) and therefore

$$a_1 \le b_1, a_2 \le b_2 \quad \Rightarrow \quad T(a_1, a_2) + T(b_1, b_2) \ge T(a_1, b_2) + T(b_1, a_2)$$

Now we prove 4), i.e.,: if $b \ge c$ then $a_1 < a_2 \Rightarrow h(a_1, b) \le h(a_2, b)$. Indeed

$$h(a_2, b) - h(a_1, b) = T(a_2, b) + T(1 - a_2, c) - T(a_1, b) - T(1 - a_1, c) = T(a_2, b) - T(a_1, b) +$$
$$+ T(1 - a_2, c) - T(1 - a_1, c) = T(a_2, b) - T(a_1, b) + c - T(a_2, c) - c + T(a_1, c) =$$
$$= T(a_1, c) + T(a_2, b) - T(a_2, c) - T(a_1, b) \ge 0$$

Analogously we can prove 5), i.e., if $b \le c$ then $a_1 < a_2 \Rightarrow h(a_1, b) \ge h(a_2, b)$.

Theorem allows to generate RET operators from convenient t-norms. It is clear that such convenient t-norm is the product, i.e. T(a,b) = ab. The technique of the construction of continuous t-norms T with a generator fulfilling T(a,c) + T(1-a,c) = c is already known [4],[1]. Moreover it is possible to construct some non continuous t-norms with given section [11]. The characterisation of such t-norm with convex generator is an open problem.

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