SMOOTHLY GENERATED DISCRETE AGGREGATION OPERATORS

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Abstract. Discrete (ordinal) scales equipped with smooth triangular norms were introduced by Godo and Sierra. Smoothly generated aggregation operators on these scales are introduced, including the only smooth generated t-norm. The influence of additional properties on the introduced operators is discussed, leading to some surprising results. Some other results are conjectured.

Key words: aggregation operator, discrete scale, generator.

1. INTRODUCTION

Continuously generated aggregation operators on the unit interval [0;1] were introduced and discussed in [11], see also [3]. For computer science applications, finite scales are more important than interval scales. Therefore, Godo and Sierra [8] introduced so called smooth triangular norms on finite chains (which can be understood also as finite ordinal scales). The structure of these t-norms was examined in [12] and it was shown that there is unique Archimedean smooth t-norm.

Aggregation operators (we will always deal with binary operators in this paper) are monotone operators (in both coordinates) such that the aggregation of minimal inputs (of maximal inputs) gives the minimal output (maximal output). Triangular norms are special aggregation operators which are symmetric, associative, and maximal element is their neutral element (for t-conorms, minimal element is their neutral element), see e.g. [4, 10].

The aim of this paper is to continue in the spirit of Godo and Sierra [8] in the class of aggregation operators which are generated by means of some generating triple. Several additional properties of these operators will give more specific results, similarly as it was in the case of smooth t-norms.

2. SMOOTHLY GENERATED DISCRETE AGGREGATION OPERATORS

Let $\mathcal{C}^* = \{x_0, \ldots, x_n\}$, $n \in N$, be a given finite (ordinal) scale. Then $\mathcal{C}^*$ can be without any loss of generality identified with the scale $\mathcal{C} = \{0, 1, \ldots, n\}$ which will be fixed since now.

Definition 1. An aggregation operator **A**: $\mathcal{C}^2 \to \mathcal{C}$ is a non-decreasing mapping such that **A**(0, 0) = 0 and **A**(n, n) = n.

The smoothness of an operator on a discrete chain introduced in [8] means, in fact, the Lipschitz property (with constant 1) on the index set.

Definition 2. An aggregation operator **A**: $\mathcal{C}^2 \to \mathcal{C}$ will be called a *smooth aggregation operator* if and only if for all a, b, c, d $\in \mathcal{C}$

$$|\mathbf{A}(a, b) - \mathbf{A}(c, d)| \leq |a - c| + |b - d| .$$

Recall that **A** is smooth iff $\mathbf{A}(a + 1, c) \leq \mathbf{A}(a, c) + 1$ and $\mathbf{A}(c, a + 1) \leq \mathbf{A}(c, a) + 1$ for all a, c $\in \mathcal{C}$, a $\neq$ n.

Following [11], we introduce the notion of a smooth generating triple.

Definition 3. Let *f*, *g*: $\mathcal{C} \to \mathcal{C}$ be two non-decreasing smooth mappings, $f(0) = g(0) = 0$ and $f(a + 1) \leq f(a) + 1$, $g(a + 1) \leq g(a) + 1$ for any a $\in \mathcal{C}$, a $\neq$ n, and $f(n) + g(n) \geq n$. Let h: $\{0, 1, \ldots, f(n) + g(n)\} \to \mathcal{C}$ be a non-decreasing surjective mapping. Then the triple (*f*, *g*, *h*) will be called a *smooth generating triple* on $\mathcal{C}$.

Note that the surjectivity and monotonicity of *h* ensures its smoothness, too.

Proposition 1. Let (*f*, *g*, *h*) be a given smooth generating triple on $\mathcal{C}$. Then the operator **A**: $\mathcal{C}^2 \to \mathcal{C}$ given by

$$\mathbf{A}(a, b) = h (f(a) + g(b)) \qquad (1)$$

is a smooth aggregation operator on $\mathcal{C}$.

The only smooth generated t-norm T on $\mathcal{C}$ is generated by the unique smooth generating triple (*f*, *f*, *h*), where $f: \mathcal{C} \to \mathcal{C}$ and $h: \{0, 1, \ldots, 2n\} \to \mathcal{C}$ are defined by $f(a) = a$ and $h(b) = \max(0, b - n)$.

Conjecture 1. If a smooth aggregation operator **A** on $\mathcal{C}$ is generated via (1) by a smooth generating triple (f, g, h) then this triple is unique.

3. SPECIAL SMOOTHLY GENERATED DISCRETE AGGREGATION OPERATORS

Now, we will examine some special properties in the class of smoothly generated discrete aggregation operators.

Symmetry.

$$\mathbf{A}(a, b) = \mathbf{A}(b, a) \qquad \text{for all } a, b \in \mathcal{C}.$$

It is evident that each smooth generating triple (f, f, h) on $\mathcal{C}$ generates a symmetric smooth aggregation operator on $\mathcal{C}$.

Conjecture 2. An aggregation operator **A** on $\mathcal{C}$ is smoothly generated and symmetric if and only if it is generated by a smooth generating triple (f, f, h).

Example 1. For n = 2k, define $f: \mathcal{C} \to \mathcal{C}$ by $f(a) = \left[\frac{a}{2}\right]$, where [x] is the integer part of the real x. Let $h: \mathcal{C} \to \mathcal{C}$ be the identity. Then (f, f, h) is a smooth generating triple on $\mathcal{C}$ and the corresponding symmetric smooth aggregation operator $\mathbf{A}: \mathcal{C}^2 \to \mathcal{C}$ is given by

$$\mathbf{A}(a, b) = \left[\frac{a}{2}\right] + \left[\frac{b}{2}\right].$$

Neutral element.

$$\mathbf{A}(a, e) = \mathbf{A}(e, a) = a \quad \text{for some } e \in \mathcal{C} \text{ and all } a \in \mathcal{C}.$$

Proposition 2. A smoothly generated aggregation operator $\mathbf{A}_{(e)}$ on $\mathcal{C}$ has the neutral element $e \in \mathcal{C}$ if and only if $\mathbf{A}_{(e)}$ is generated by the smooth generating triple (f, f, $h_{(e)}$), where $f: \mathcal{C} \to \mathcal{C}$ is the identity and $h_{(e)}: \{0, 1, \dots, 2n\} \to \mathcal{C}$ is given by

$$h_{(e)} = \max(0, \min(b - e, n)).$$

Note that the existence of a neutral element forces the symmetry of $\mathbf{A}_{(e)}: \mathcal{C} \to \mathcal{C}$, which is given by

$$\mathbf{A}_{(e)}(a, b) = h_{(e)}(a + b) = \max(0, \min(a + b - e, n)),$$

compare similar operators on [0`; 1] introduced in [13]. It is straightforward that $\mathbf{A}_{(e)}$ is associative only if e = 0 (and then $\mathbf{A}_{(e)} = \mathbf{A}_{(0)}$ is the only smooth Archimedean t–conorm on $\mathcal{C}$) or e = n (and then $\mathbf{A}_{(e)} = \mathbf{A}_{(n)}$ is the only smooth Archimedean t-norm on $\mathcal{C}$).

Annihilator (zero element).

$$\mathbf{A}(z, a) = \mathbf{A}(a, z) = z \quad \text{for some } z \in \mathcal{C} \text{ and all } a \in \mathcal{C}.$$

Proposition 3. A smoothly generated aggregation operator $\mathbf{A}^{(z)}$ on $\mathcal{C}$ has an annihilator $z \in \mathcal{C}$ if and only if $\mathbf{A}^{(z)}$ is generated by the smooth generating triple (*f*, *f*, $h^{(z)}$), where $f: \mathcal{C} \to \mathcal{C}$ is identity and $h^{(z)}: \{0, 1, \ldots, 2n\} \to \mathcal{C}$ is given by

$$h^{(z)}(b) = \begin{cases} b & \text{if } b \le z \\ z & \text{if } z \le b \le n + z \\ b - z & \text{if } n + z \le b \end{cases}.$$

Again the existence of an annihilator forces the symmetry of $\mathbf{A}^{(z)} : \mathcal{C}^2 \to \mathcal{C}$ which is a version of generalized median, compare [1, 2, 5, 7],

$$\mathbf{A}^{(z)}(a, b) = \text{med}(a, b, z).$$

For any $z \in \mathcal{C}$, the operator $\mathbf{A}^{(z)}$ is also associative. Note that $\mathbf{A}^{(0)} = \mathbf{A}_{(1)}$ and $\mathbf{A}^{(1)} = \mathbf{A}_{(0)}$.

Associativity.

$$\mathbf{A}(\mathbf{A}(a, b), c) = \mathbf{A}(a, \mathbf{A}(b, c)) \quad \text{for all } a, b, c \in \mathcal{C}.$$

We have already seen that the operators $\mathbf{A}^{(z)}$ are associative.

Conjecture 3. A smoothly generated aggregation operator **A** on $\mathcal{C}$ is associative if and only if

i. **A** is symmetric and then $\mathbf{A} = \mathbf{A}^{(z)}$ for some $z \in \mathcal{C}$;

ii. **A** is non-symmetric and then either **A**(a, b) = b for all $a, b \in \mathcal{C}$, or **A**(a, b) = a for all $a, b \in \mathcal{C}$, i. e., **A** is a projection operator.

Note that the associativity of aggregation operators on [0; 1] generated by means of generating triples was examined in [3], compare also [6].

4. CONCLUSIONS

The continuity of operators on interval scales can be understood as the smoothness in the case of discrete scales (compare, e. g., the intermediate value property). Consequently, we can expect an increasing interest about the smooth aggregation operators on discrete scales, see, e. g., [9]. We have introduced and discussed a distinguished class of such operators– smoothly generated aggregation operators. We have addressed several open problems in the form of conjectures. We expect also that several other properties of smoothly generated aggregation operators will be studied, e. g., idempotency, bisymmetry, compensation property, etc.

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