ON THE COMPARISON OF QUASI-ARITHMETIC MEANS

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ABSTRACT. Necessary and sufficient conditions for the comparison of quasi-arithmetic means by help of their generators are proved. Moreover, the equality of two quasi-arithmetic means is discussed. Several examples are given.

1. INTRODUCTION

Quasi-arithmetic means are transformations of the common arithmetic mean. In this paper we will consider the extended arithmetic mean

$$M : \bigcup_{n\in\mathbb{N}} \bar{\mathbb{R}}^n \to \bar{\mathbb{R}}, \quad M(u_1, ..., u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i.$$

If $\{-\infty, \infty\} \subseteq \{u_1, ..., u_n\}$, we will use the convention $-\infty + \infty = -\infty$.

Definition 1. Let $f : [0,1] \to \bar{\mathbb{R}}$ be a continuous strictly monotone mapping. A quasi-arithmetic mean is an operator $M_f : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ defined by

$$M_f(x_1, ..., x_n) = f^{-1}\left(M\left(f(x_1), ..., f(x_n)\right)\right), \tag{1}$$

where f^{-1} is the inverse function of f.
The function f is called the generator of the quasi-arithmetic mean M_f.

Directly from the definition it follows that $M_{af+b} = M_f$ for each a, $b \in \mathbb{R}$, $a \neq 0$.

Quasi-arithmetic means belong to the class of aggregation operators. Without loss of generality we can consider the aggregation operators with inputs from the unit interval. Our definition of an aggregation operator agrees, e.g., with definitions given in [8,6].

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Definition 2. A mapping $A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ is an aggregation operator if it is monotone, i.e.,

$$A(x_1, ..., x_n) \le A(y_1, ..., y_n) \text{ whenever } x_i \le y_i \text{ for each } i = 1, ..., n \text{ and } n \in \mathbb{N},$$

and

if for each $x \in [0,1]$ and $n \in \mathbb{N}$ the following boundary conditions are satisfied :

(A1) $A(x) = x$,

(A2) $A(\underbrace{0, ..., 0}_{n\text{-times}}) = 0$,

(A3) $A(\underbrace{1, ..., 1}_{n\text{-times}}) = 1$.

2. Comparison of quasi-arithmetic means

Definition 3. We say that an aggregation operator A is weaker than an aggregation operator B and denote $A \le B$, if for all possible inputs $(x_1, ..., x_n) \in [0,1]^n$, $n \in \mathbb{N}$, the inequality $A(x_1, ..., x_n) \le B(x_1, ..., x_n)$ holds.

Although this definition is given for any two aggregation operators, in this section we prove only necessary and sufficient conditions for comparing quasi-arithmetic means.

It is well-known that harmonic , geometric, arithmetic and quadratic means are comparable in this sense and $H \le G \le M \le Q$.

Triangular norms [3] are special types of aggregations operators. For the basic four t-norms, i.e., Drastic product T_D, Lukasiewicz t-norm T_L, Product T_P and Minimum T_M we have $T_D \le T_L \le T_P \le T_M$. Further, it is known [5] that if t-norms T_1,T_2 are generated by continuous additive generators f_1 and f_2, respectively, then $T_1 \le T_2$ iff $f_1 \circ f_2^{-1}$ is a subadditive function, i.e., $f_1 \circ f_2^{-1}(x+y) \le f_1 \circ f_2^{-1}(x) + f_1 \circ f_2^{-1}(y)$ for all x, y, $x+y \in$ Ran f_2.

A similar characterization by means of generators can be also stated for quasi-arithmetic means .

Theorem 1. Let M_f be a quasi-arithmetic mean generated by an increasing (decreasing) generator f and let M_g be any quasi-arithmetic mean. Then M_f is weaker then M_g if and only if the composite function $f \circ g^{-1}$ is concave (convex) on the *Ran* g.

Proof. The property $M_f \le M_g$ is equivalent to

$$f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) \le g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right) \tag{2}$$

for all n-tuples $(x_1, ..., x_n) \in [0,1]^n$, $n \in \mathbf{N}$.

Let f be an increasing function. Then Eq.(2) is equivalent to

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \leq f \circ g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right).$$

Putting $g(x_i) = u_i$, $i = 1, ..., n$, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} f \circ g^{-1}(u_i) \leq f \circ g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} (u_i)\right)$$

for each $u_i \in Ran\ g$, $i = 1, ..., n$ and $n \in \mathbf{N}$. However, the last inequality is equivalent to the concavity of the composite function $f \circ g^{-1}$.

The proof for a decreasing generator runs as before.

Example 1. The decreasing generators of the harmonic , geometric, arithmetic and quadratic means are functions given by

$$f_H(x) = \frac{1-x}{x}, \qquad f_G(x) = -\log x,$$

$$f_M(x) = 1 - x, \qquad f_Q(x) = 1 - x^2.$$

As it is easily seen, all composites

$$f_H \circ f_G^{-1}(x) = \exp x - 1, \quad f_G \circ f_M^{-1} = -\log(1-x), \quad f_M \circ f_Q^{-1}(x) = 1 - \sqrt{1-x}$$

are convex which proves the inequality

$$H \leq G \leq M \leq Q.$$

Example 2. Consider the operator M_f defined by :

$$M_f(x_1, ..., x_n) = \frac{\left(\prod_{i=1}^{n} x_i\right)^{1/n}}{\left(\prod_{i=1}^{n} x_i\right)^{1/n} + \left(\prod_{i=1}^{n} (1-x_i)\right)^{1/n}} \tag{3}$$

if $\{0,1\} \not\subseteq \{x_1, ..., x_n\}$ and $M_f(x_1, ..., x_n) = 0$, otherwise.

It can be shown that this operator is the quasi-arithmetic mean generated by the generator $f(x) = \log \frac{x}{1-x}$, whose inverse function is given by

$$f^{-1}(x) = \frac{\exp x}{1 + \exp x}.$$

Further, consider the geometric mean whose generator $f_G(x) = -\log x$ is a decreasing function. Since the composite function

$$f_G \circ f^{-1}(x) = \log(1 + \exp x) - x$$

is convex, we obtain $G \le M_f$, which can be also seen from Eq.(3).

Remark 1. The operator M_f introduced in Example 2 has the annihilator $a = 0$ and the weak annihilator $b = 1$, see [7]. Moreover, it can be derived from the conjunctive associative compensatory operator C [1,4] generated by f that is defined by

$$C(0,1) = C(1,0) = 0 \text{ and } C(x,y) = \frac{xy}{xy + (1-x)(1-y)}, \text{ otherwise}$$

in an iterative way described in [7].

Note that the quasi-arithmetic mean M_f mentioned in Example 2 and arithmetic mean M are not comparable. For instance, $M_f(0, 0.8) = 0 < 0.4 = M(0, 0.8)$ and $M_f(0.2, 1) = 1 > 0.6 = M(0.2, 1)$.

The composite functions are neither convex nor concave.

Corollary. Let M_f, M_g be quasi-arithmetic means . Then $M_f = M_g$ if and only if $f \circ g^{-1}$ is a linear function, i.e., $g = af + b$ for some $a, b \in \mathbb{R}$, $a \neq 0$.

Proof. As it was already mentioned the first part of the claim, i.e., $M_{af+b} = M_f$ for each generator f and each a, $b \in \mathbb{R}$, $a \neq 0$, follows directly from the definition of M_f. From this it is also clear that for each quasi-arithmetic mean an increasing generator can be chosen (using a linear transformation if necessary).

Let M_f, M_g be two quasi-arithmetic means for which $M_f = M_g$. Without loss of generality we can assume that M_f and M_g are generated by increasing generators. Then from $M_f \le M_g$ and $M_g \le M_f$ we obtain $f \circ g^{-1}$ and $g \circ f^{-1}$ are concave functions.

Since $g \circ f^{-1} = \left(f \circ g^{-1}\right)^{-1}$, $g \circ f^{-1}$ has to be a convex function. Thus $g \circ f^{-1}$ is a linear function (non-trivial) and consequently, $g = af + b$ for some a, $b \in \mathbb{R}$, $a > 0$. The situation is similar for decreasing generators. It can be shown that $a < 0$ corresponds to the case when one of the given generators is increasing and the other decreasing.

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