# ON SHARP ELEMENTS IN LATTICE ORDERED EFFECT ALGEBRAS

GEJZA JENČA AND ZDENKA RIEČANOVÁ

ABSTRACT. In the present paper, we show that in a lattice ordered effect algebra $E$ the set of all sharp elements forms an orthomodular lattice whose blocks are centers of blocks of $E$ and whose center coincides with the center of the effect algebra $E$. Moreover, we extend the well-known result about connection between compatibility and distributivity from the theory of orthomodular lattice to the class of lattice ordered effect algebras.

## 1. INTRODUCTION

**Definition 1.1.** [2] An *effect algebra* is a partial algebra $E$ with partial binary operation $\oplus$ and two nullary operations 0 and 1 satisfying the following axioms:

(E1) $a \oplus b = b \oplus a$ if $a \oplus b$ is defined.
(E2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined.
(E3) for every $a \in E$ there exists a unique $b \in E$ such that $a \oplus b = 1$.
(E4) if $1 \oplus a$ is defined then $a = 0$.

Having an effect algebra $E$, we can introduce a partial order $\leq$ on $E$ : $a \leq b$ iff $\exists c : a \oplus c = b$. We denote $b \ominus a = c$ iff $a \oplus c = b$. It is easy to check that $\ominus$ is a well defined partial operation. In [7], a class of partial structures equivalent to effect algebras, so-called *D-posets*, was introduced independently. The axioms for D-posets are based on $\ominus$.

As usual, we denote $1 \ominus x$ by $x'$. Further, we denote $a \perp b$ iff $a \oplus b$ exists iff $a \leq b'$ iff $b \leq a'$. If $E$ is an effect algebra and $(E, \leq)$ is a lattice, then $E$ is called *lattice ordered.* Every lattice ordered effect algebra satisfies the De Morgan law : $(a \vee b)' = a' \wedge b'$. Lattice ordered effect algebras are called *D-lattices* in [8].

Examples of lattice ordered effect algebras are :

- Any horizontal sum of two MV-algebras (c.f. [1]).
- Any horizontal sum of an OML and an MV-algebra.

1991 *Mathematics Subject Classification.* Primary 06F05; Secondary 03G25,81P10.
*Key words and phrases.* effect algebra, MV-algebra, orthomodular lattice.
This research is supported by grant G-1/4297/97 of the MŠ SR, Slovakia.

- Any direct product of an OML and an MV-algebra.

In an effect algebra $E$, a pair of elements $a, b$ is called *compatible* ($a \leftrightarrow b$) iff there are $a_1, b_1, c \in E$ such that $a_1 \oplus b_1 \oplus c$ exists and $a = a_1 \oplus c$, $b = b_1 \oplus c$. In a lattice ordered effect algebra $E$, we have $a \leftrightarrow b$ iff $(a \vee b) \ominus b = a \ominus (a \wedge b)$ (c.f. [8]) iff $a \oplus (b \ominus (a \wedge b))$ exists (c.f. [10]).

In accordance with [10], we define a *block* in $E$ as a maximal mutually compatible subset of $E$. Every element of $E$ is in some block. By [7] and [8], an *MV-algebra* (introduced by Chang in [1]). can be defined as a lattice ordered effect algebra of mutually compatible elements.

In [10], it was proved that in a lattice ordered effect algebra $E$, every block $M$ of $E$ has the following properties.

(1) $a, b \in M$ implies that $a \wedge b, a \vee b \in M$.
(2) $a, b \in M$, $a \perp b$ imply that $a \oplus b \in M$.
(3) $a \in M$ implies $a' \in M$.
(4) $M$ is an MV-algebra.

We note that (2) and (3) mean that every block $M$ is a sub-effect algebra of $E$. In particular, $a, b \in M$ with $a \geq b$ implies that $a \ominus b \in M$.

## 2. Compatibility and distributivity

**Theorem 2.1.** *Let $E$ be a lattice ordered effect algebra. Assume $b \in E$, $A \subseteq E$ are such that $\vee A$ exists in $E$ and $b \leftrightarrow a$ for all $a \in A$. Then*

(a) $b \leftrightarrow \vee A$.
(b) $\vee\{b \wedge a : a \in A\}$ *exists in $E$ and equals* $b \wedge (\vee A)$.

*Proof.*

(a) For every $a \in A$,

$$a \leq (b \ominus (a \wedge b))' \leq (b \ominus ((\vee A) \wedge b))'$$

Therefore,

$$\vee A \leq (b \ominus ((\vee A) \wedge b))'$$

and this is equivalent to $\vee A \leftrightarrow b$.

(b) First, let us prove

$$\wedge \{b \ominus (b \wedge a) : a \in A\} = b \ominus (b \wedge (\vee A)) \tag{1}$$

Assume $d$ is a lower bound of $\{b \ominus (b \wedge a) : a \in A\}$. Since $b \leftrightarrow a$, we have $b \ominus (b \wedge a) = (b \vee a) \ominus a$. Evidently, $d \leq (b \vee (\vee A)) \ominus a$ for all $a \in A$. This implies $a \leq (b \vee (\vee A)) \ominus d$ and we have $\vee A \leq (b \vee (\vee A)) \ominus d$ which is equivalent to $d \leq (b \vee (\vee A)) \ominus (\vee A)$. By part (a), $b \leftrightarrow \vee A$. Therefore,

$$d \leq (b \vee (\vee A)) \ominus (\vee A) = b \ominus (b \wedge (\vee A))$$

which completes the proof of (1).

Now, let $e$ be an upper bound of $\{b \wedge a : a \in A\}$. For all $a \in A$, $b \wedge a \leq b \wedge e$ and $b \ominus (b \wedge e) \leq b \ominus (b \wedge a)$. Therefore, $b \ominus (b \wedge e) \leq \wedge\{b \ominus (b \wedge a) : a \in A\}$. Applying (1), $b \ominus (b \wedge e) \leq b \ominus (b \wedge (\vee A))$. Thus, $b \wedge (\vee A) \leq (b \wedge e) \leq e$. Moreover, $b \wedge (\vee A)$ is an upper bound of $\{b \wedge a : a \in A\}$. This completes the proof.

□

**Corollary 2.2.** *Let $E$ be a lattice ordered effect algebra. Assume $b \leftrightarrow a_1$, $b \leftrightarrow a_2$. Then $b \wedge (a_1 \vee a_2) = (b \wedge a_1) \vee (b \wedge a_2)$.*

*Proof.* Trivial, put $A = \{a_1, a_2\}$ in part (b) of Theorem 2.1. □

## 3. Blocks and sharp elements

An element $a$ of an effect algebra is called *sharp* iff $a \wedge a' = 0$. The set of all sharp elements in an effect algebra $E$ is denoted by $E_S$.

An element $a$ of an effect algebra $E$ is called *central* iff

(1) The intervals $[0, a]$ and $[0, a']$ are closed on $\oplus$ operation, i.e. $x, y \in [0, b]$ and $x \perp y$ imply $x \oplus y \leq b$, where $b \in \{a, a'\}$.

(2) Every $x \in E$ has a unique decomposition $x = x_1 \oplus x_2$, where $x_1 \in [0, a]$ and $x_2 \in [0, a']$.

The set of all central elements of $E$ (the *center of $E$*) is denoted by $C(E)$. There is a natural, one-to-one correspondence between central elements and direct decompositions of $E$. The center of every effect algebra forms a sub-effect algebra, which is a Boolean algebra in its own right. We refer to [5] for further results concerning central elements.

**Lemma 3.1.** *Let $E$ be a lattice ordered effect algebra, $a \in E$. The following are equivalent :*

(a) *$a$ is sharp.*

(b) *$a$ is central in every block containing $a$.*

(c) *$a$ is central in some block.*

*Proof.* ((a) $\Longrightarrow$ (b)) Assume $a$ is sharp. Let $M$ be a block of $E$ such that $a \in M$. Evidently, $a$ is sharp in $M$. By, [9], an element $x$ of an lattice ordered effect algebra is central iff $x$ is sharp and $x$ is compatible with every element. Since every pair of elements in $M$ is compatible, this implies that $a$ is central in $M$.

((b) $\Longrightarrow$ (c)) is trivial.

((c) $\Longrightarrow$ (a)) Assume $a$ is central in some block $M$. Then $a$ is sharp in $M$ and, since $M$ is a sublattice of $E$, $a$ is sharp in $E$. □

**Corollary 3.2.** *Let $E$ be a lattice ordered effect algebra.*

$$E_S = \cup\{C(M) : M \text{ is a block of } E\}$$

**Theorem 3.3.** *Let $E$ be a lattice ordered effect algebra. Then*

(a) *$E_S$ is a sub-effect algebra of $E$.*
(b) *$E_S$ is a sublattice of $E$.*
(c) *$E_S$ is an orthomodular lattice.*

*Proof.*

(a) Assume $a, b \in E_S$, $a \perp b$. As $a \perp b$, we have $a \leftrightarrow b$ so $a, b \in M$ for some block $M$. By Lemma 3.1, $a$ and $b$ are central in $M$. Since $C(M)$ is a subalgebra of $E$, we have $a \oplus b \in C(M) \subseteq E_S$. It remains to observe that $a \in E_S$ implies $a' \in E_S$.

(b) Assume $a, b \in E_S$. Note that $a' \perp a \wedge b$ and $b' \perp a \wedge b$. This implies $a' \leftrightarrow a \wedge b$ and $b' \leftrightarrow a \vee b$. Applying Corollary 2.2, we have

$$(a \wedge b) \wedge (a \wedge b)' = (a \wedge b) \wedge (a' \vee b') = ((a \wedge b) \wedge a') \vee ((a \wedge b) \wedge b') = 0$$

Thus, $a \wedge b$ is sharp. Similarly, $a, b \in E_S$ implies $a \vee b \in E_S$.

(c) For all $x, y \in E$ we have $(x')' = x$ and $x \leq y \implies y' \leq x'$. For $x, y \in E_S \subseteq E$ we have $x \wedge x' = 0$. Thus $E_S$ is an ortholattice. It remains to prove the orthomodular law.
Assume $x, y \in E_S$, $x \leq y$. Since $x' \perp x$, $x' \leftrightarrow x$. Since $x \leq y$, we have $y' \leq x'$ and this implies $x' \leftrightarrow y'$. Applying Corollary 2.2,

$$x' \wedge (x \vee y') = (x' \wedge x) \vee (x' \wedge y') = 0 \vee y' = y'$$

which implies $y = x \vee (x' \wedge y)$, hence the orthomodular law is satisfied.

□

**Lemma 3.4.** *Let $E$ be a lattice ordered effect algebra, $a, b \in E_S$. Then $a$ and $b$ are compatible in $E$ iff they are compatible in $E_S$.*

*Proof.* Assume $a, b \in E_S$. By [8], in a D-lattice $a \leftrightarrow b$ iff $a \ominus (a \wedge b) = (a \vee b) \ominus b$. By above theorem, this equation is true in $E$ iff it is true in $E_S$. □

**Theorem 3.5.** *Let $E$ be a lattice ordered effect algebra. An element $a$ of $E$ is central iff $a$ is central in every block of $E$.*

*Proof.* Assume $a$ is central in $E$. It is easy to check that $a$ is compatible with every element of $E$. Thus, $a$ is in every block of $E$. By Lemma 3.1, $a$ is central in every block of $E$ since $a \in E_S$.

Assume that $a$ is central in every block of $E$. Since $a$ is in every block of $E$, we see that $a$ is compatible with every element of $E$. By [9], an element $x$ of a lattice ordered effect algebra $E$ is central iff $x$ is sharp and $x \leftrightarrow y$ for all $y \in E$. Thus, $a$ is central. □

**Theorem 3.6.** *Let $E$ be a lattice ordered effect algebra. Let $C$ be a maximal pairwise compatible subset of $E_S$. Then $C = C(M)$ for some block $M$ of $E$. Moreover, for every block $M$ of $E$ with $M \supseteq C$, $C = C(M)$.*

*Proof.* By Lemma 3.4, $C$ is pairwise compatible in $E$. Hence there is some block $M \supseteq C$ in $E$. Evidently, $C \subseteq C(M)$. On the other hand, assume $a \in C(M)$. Then $a \in E_S$ and $a \leftrightarrow c$ for all $c \in C$. By maximality of $C$, $a \in C$. □

Recall that $F$ is a *full sublattice* of a (not necessary complete) lattice $L$ iff $F$ is a sublattice which contains all existing joins and meets of elements of $F$.

**Theorem 3.7.** *Let $E$ be a lattice ordered effect algebra. Then $E_S$ is a full sublattice of $E$.*

*Proof.* Let $A \subseteq E_S$ be such that $\vee A$ exists in $E$. We have to show that $\vee A \in E_S$. Observe that $(\wedge_{a\in A} a') \leftrightarrow a'$ for all $a \in A$. Hence $(\wedge_{a\in A} a') \leftrightarrow a$ for all $a \in A$. By Theorem 2.1, we have

$$\begin{aligned}(\vee A) \wedge (\vee A)' &= (\vee_{a\in A} a) \wedge (\wedge_{a\in A} a') = \\ &= \vee_{a\in A}(a \wedge (\wedge_{a\in A} a')) \leq \vee_{a\in A}(a \wedge a') = 0\end{aligned}$$

Thus $\vee A \in E_S$.

Dually, $\wedge A \in E_S$. □

**Corollary 3.8.** *Let $E$ be an effect algebra such that $(E, \leq)$ is a complete lattice. Then $E_S$ is a complete lattice.*

*Proof.* Trivial, by Theorem 3.7. □

**Remark 3.9.** Assume $E$ is a lattice ordered effect algebra.

(1) Evidently, $E = E_S$ iff $E$ is an orthomodular lattice (under the orthocomplementation $a' = 1 \ominus a$ and a partial operation $\oplus$ defined by $a \oplus b = a \vee b$ iff $a \leq b'$.
(2) If $E$ is a horizontal sum of two MV-algebras then $E_S$ is a horizontal sum ($0-1$-pasting) of two Boolean algebras.
(3) If $E$ is a direct product of an orthomodular lattice $L$ and an MV-algebra $M$, then $E_S$ is the direct product of $L$ and a Boolean algebra $C(M)$.
(4) If $E$ is a block-finite lattice ordered effect algebra then $E_S$ is a block finite orthomodular lattice – this is an obvious consequence of Theorem 3.6. The converse does not hold.

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Department of Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Ilkovičova 3, 812 19 Bratislava, Slovakia

*E-mail address*, G. Jenča: jenca@kmat.elf.stuba.sk

*E-mail address*, Z. Riečanová: zriecan@.elf.stuba.sk