NEARNESS, CONVERGENCE AND TOPOLOGY

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ABSTRACT. The aim of this article is to discuss a type of convergence, compactness and continuity based on the concept of a nearness.

In the first part we restrict our attention to the real case. Section 4 presents a way how to generalize the nearness for an arbitrary universe.

1. PRELIMINARIES

The purpose of this paper is to continue the study of a fuzzification of metric properties of real numbers, based on the concept of a nearness, investigated, for example, already in papers [2], [3], [4] and [5]. In [2] is the nearness introduced as a binary fuzzy relation N on $\mathbf{R}$ with some natural properties, corresponding to the algebraic, topological and lattice structures of real axis:

(1) $N(x,x) = 1$ for each $x \in \mathbf{R}$

(2) $N(x,y) = N(y,x)$ for each $x, y \in \mathbf{R}$

(3) $N(x,y) \geq N(x,z)$ for each $x, y, z \in \mathbf{R}$, such that $x \leq y \leq z$

(4) $\lim_{n\to\infty} x_n = \infty \Longrightarrow \lim_{n\to\infty} N(x_n, x_0) = 0$, for each $x_0 \in \mathbf{R}$

(5) $N(x,y) = N(x+z, y+z)$ for each $x, y, z \in \mathbf{R}$

By [2] a binary fuzzy relation $N(x,y)$ on $\mathbf{R}$ is the nearness if and only if there exists such a non-increasing function $b : [0,\infty] \to [0,1]$ such that $b(0) = 1$, $\lim_{x\to\infty} b(x) = 0$ and

$$N(x,y) = b(|x-y|) \quad \text{for each} \quad x, y \in \mathbf{R}.$$

This function b is determined uniquely and it is called the nearness-generating function. Compare also [1] and [6].

In the following article we utilize just this approach.

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Further, let us recall, that a sequence of real numbers $\{x_n\}$ is called N-convergent and a real number x_0 is called its N-limit ($x_n \underset{N}{\rightarrow} x_0$) if

$$\lim_{n\to\infty} N(x_n, x_0) = 1.$$

This N-convergence enables to define an N-continuity for real functions of a real variable [2]:

Suppose $x_0 \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be N-continuous at x_0 if for each $\epsilon < 1$ there exists $\delta < 1$ such that for each $x \in \mathbb{R}$:

$$N(x_0, x) > \delta \Longrightarrow N(f(x_0), f(x)) > \epsilon.$$

As usual, a function is said to be continuous on a set $A \subset \mathbb{R}$ if it is continuous at each $x \in A$.

And finally, if for each $\epsilon < 1$ there exists $\delta < 1$ such that for each $x, y \in A \subset \mathbb{R}$:

$$N(x, y) > \delta \Longrightarrow N(f(x), f(y)) > \epsilon,$$

the function f is said to be uniformly N-continuous on A.

It can be shown (see [2]), that a function f is N-continuous at a point x_0 just if for any sequence $\{x_n\}$ of real numbers

$$\lim_{n\to\infty} N(x_n, x_0) = 1 \Longrightarrow \lim_{n\to\infty} N(f(x_n), f(x_0)) = 1.$$

It is evident, that the family of N-convergent sequences, and therefore also the family of N-continuous and uniformly N-continuous functions depends substantively on the concrete nearness and thus on the nearness-generating function b.

2. N-CONVERGENCE IN $\mathbb{R}$

N-convergence or N-divergence is, in fact, influenced only by the behaviour of the corresponding nearness-generating function b in a right neighbourhood of 0.

From this point of view we can divide the set of all nearness-generating functions into 3 subsets:

(a) $b(x) \to 1^- \iff x \to 0^+$. (In other words: $b(0^+) = 1$ and $b(x) > 1$ whenever $x > 0$.)

(b) There exists $K < 1$, such that for each $x > 0$ is $b(x) < K$.

(c) There exists $K > 0$, such that for each $x < K$ is $b(x) = 1$.

It can be easily seen, that in the case (a)

$$x_n \underset{N}{\rightarrow} x_0 \iff |x_n - x_0| \to 0.$$

It follows, that this N-convergence and the convergence with respect to the standard metric of real numbers are identical. The property (a) is obviously fulfilled, for example, if b is continuous and strictly decreasing.

In the case (b) the only N-convergent sequences are sequences stationary from a term (see [3]), thus, these nearness-generating functions give a trivial type of N-convergence.

Finally, a nearness in the case (c) does not distinguish points distant less than K and as a consequence, N-convergence is a little unusual and is affected also by the right continuity or discontinuity of b at the maximal point K with the above mentioned property.

As we can simply prove, any fundamental sequence of real numbers is under the condition (c) N-convergent and has infinitely many N-limits.

Of course, not only the fundamental sequences are N-convergent:

Example 1. Let

$$b(x) = \begin{cases} 1, & \text{for } x \leq 1, \\ 0, & \text{for } x > 1. \end{cases} \quad \text{and} \quad N(x,y) = b(|x-y|) \text{ for } x, y \in \mathbb{R}.$$

Consider now the sequence $\{x_n\} = \{0, 2, 0, 2, ...\}$. Despite the sequence is neither fundamental nor N-fundamental in a natural sence (because $N(x_n, x_{n+1}) = 0$, for each $n \in \mathbb{N}$), it is N-convergent and it has just one limit, the number 1.

The problem of N-continuity, studied in [3], is closely connected with the foregoing discussed N-convergence.

It is shown there, that if the nearness-generating function b of a nearness N has the property (a), then a real function of a real variable is N-continuous at a point if and only if it is continuous at that point.

In the case (b) any function is N-continuous at any point.

What about the N-continuity in the case (c), in the paper [3] there are examples showing, that functions continuous at a point need not be N-continuous at the point and vice versa.

3. N-COMPACTNESS IN $\mathbb{R}$

A possible way, how to validate, that a generalization or fuzzification of classical mathematical notions is a reasonable one, is to show, that at least some of important properties, assertions and relations from the classical case are preserved.

We are familiar with the well known fact in real analysis, that continuity on a compact set implies uniform continuity.

Therefore a question arises, whether there is an analogy for N-continuity, uniform N-continuity and N-compactness.

Definition 1. Let N be a nearness. A set $A \subset \mathbb{R}$ is called an N-compact, if any sequence $\{x_n\}, x_n \in A$ for each $n \in \mathbb{N}$ contains a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \underset{N}{\rightarrow} x_0 \in A$.

Evidently, the family of N-compacts is determined by the nearness and thus by the nearness-generating function and it is different in the cases (a), (b), (c), distinguished above.

Theorem 1. *Let N be a nearness with the nearness-generating function fulfilling (a). A set $A \subset \mathbb{R}$ is an N-compact if and only if it is a compact.*

Proof. The assertion follows immediately from the fact, that the nearness establishes the same convergence, as usual metric of real numbers.

Corollary. *Let N be a nearness with the nearness-generating function satisfying (a). Let $A \subset \mathbb{R}$ be an N-compact. Then any real valued function which is N-continuous on A is uniformly N-continuous on A.*

Theorem 2. *Let N be a nearness with the nearness-generating function fulfilling (b). A set $A \subset \mathbb{R}$ is an N-compact, if and only if it is a finite set.*

Proof. The "if" part is trivial.

Suppose that $A \subset \mathbb{R}$ contains infinitely many elements. Then A obviously contains sequences not containing any stationary subsequence, hence A is not N-compact.

In [3], there is proved (Theorem 3), that in the case (b) all real functions of a real variable are uniformly N-continuous on each set $A \subset \mathbb{R}$.

Theorem 3. *Let N be a nearness with the nearness-generating function satisfying (c). A set $A \subset \mathbb{R}$ is an N-compact, if and only if it is a bounded set.*

Proof. Let $A \subset \mathbb{R}$ be bounded and let $\{x_n\}$ be a sequence of its elements. Then there exists a subsequence $\{x_{n_k}\}$ which is fundamental. Hence there exists a positive number n_0 such that $n_{k_1}, n_{k_2} > n_0$ implies $|x_{n_{k_1}} - x_{n_{k_2}}| < K$. Therefore, any element $x_{n_{k_0}}$ for $n_{k_0} > n_0$ is an N-limit of the chosen subsequence and thus A is an N-compact.

The proof of the opposite implication is indirect: Let $A \subset \mathbb{R}$ be a set, unbounded from above.(The proof for unboundedness from below is analogical.)Let $x_1 \in A$ be arbitrary. Let $K_0 > 0$ be maximal number such that $b(x) = 1$, for $x \in [0, K_0)$. Now choose $x_2 \in A$ so that $x_2 > x_1 + K_0$, then $x_3 \in A$, $x_3 > x_2 + K_0$, and so on, for each natural n there exists $x_{n+1} \in A$, $x_{n+1} > x_n + K_0$, $x_n \in A$.

It can be easily seen, that in this way chosen sequence $\{x_n\}$ cannot contain any N-convergent subsequence.

The problem, if in this case any function, N-continuous on an N-compact set is also uniformly N-continuous on this set is not yet solved, but a negative answer to this question seems to be more probable.

4. Nearness on a set

Any "reasonable" nearness on the set of all real numbers should be evidently compatible with the structure of real axis. But if X is an arbitrary set without any structure, we are not restricted by any additional requirements and conditions and a nearness in X should comply only with our intuitive idea of a nearness.

We will consider the properties of the nearness N defined as follows.

Definition 2. Let X be a set. A binary fuzzy relation N on X is called a nearness on X, if:

(N1) $N(x,x) = 1$, for each $x \in X$

(N2) $N(x,y) = N(y,x)$, for each $x, y \in X$

(N3) For each $\epsilon > 0$ there exists $\delta < 1$ such that

$$N(x,y) > \delta \Longrightarrow |N(x,z) - N(y,z)| < \epsilon, \text{ for each } x, y, z \in X.$$

The properties (N1) and (N2) , it means the reflexivity and the symmetricity of N are immediate. The property (N3) substitutes, in a sence, the triangular inequality and it has the following meaning: If two points x and y are sufficiently near one another, then the difference of their nearnesses to any other point z is arbitrarily small.

In the real case, if $X = \mathbb{R}$, it is trivial, that all nearnesses fulfill the properties (N1) and (N2).It is easily seen, that the nearness of the type (b) fullfils also the property (N3): For each $\epsilon > 0$, it is sufficient to take $\delta > K$.

As the following example shows, no nearness of the type (c) satisfies (N3).

Example 2. Let $K > 0$ be such that for $x < K$ is $b(x) = 1$ and for $x > K$ is $b(x) < 1$, and let $N(\frac{-2K}{3}, \frac{2K}{3}) = b(\frac{4K}{3}) = k_0 < 1$. Put $\epsilon_0 = 1 - k_0 < 1$.

Then despite of the fact, that $N(\frac{2K}{3}, 0) = b(\frac{2K}{3}) = 1 > \delta$, for each $\delta < 1$,

$$|N(\frac{-2K}{3}, \frac{2K}{3}) - N(\frac{-2K}{3}, 0)| = |b(\frac{4K}{3}) - b(\frac{2K}{3})| = 1 - k_0 = \epsilon_0.$$

Proposition 1. *Let $N(x,y) = b(|x-y|)$ for each $x, y \in \mathbb{R}$ be a nearness on $\mathbb{R}$ of the type (a) and let moreover the function b be continuous. Then for all $x, y, z \in \mathbb{R}$ and for each $\epsilon > 0$ there exists $\delta < 1$, such that*

$$b(|x-y|) > \delta \Longrightarrow |b(|x-z|) - b(|y-z|)| < \epsilon,$$

it means, that N satisfies the property (N3).

Proof. By assumptions it follows obviously, that the function b is uniformly continuous. Let $\epsilon_0 > 0$ be arbitrary.

Uniform continuity of b implies, that there is $\delta_0 > 0$ such that for each $x_1, x_2 \in \mathbb{R}$,

$$|x_1 - x_2| < \delta_0 \Longrightarrow |b(x_1) - b(x_2)| < \epsilon_0.$$

Put $\delta_1 = b(\delta_0) < 1$.

Now, if for a couple of real numbers x, y it holds $b(|x-y|) > \delta_1$, then $|x-y| < \delta_0$ and from the triangular inequality it follows that for any real z

$$||x-z| - |y-z|| \le |x-y| < \delta_0.$$

Therefore $|b(|x-z|) - b(|y-z|)| < \epsilon_0$.

As the following example shows, the assumption of continuity from the foregoing theorem cannot be dropped.

Example 3. Let a nearness-generating function be defined by

$$b(x) = \begin{cases} 1 - \frac{x}{10}, & \text{for } x < 5, \\ 0, & \text{for } x \ge 5. \end{cases}$$

It is easily checked, that for any $\delta < 1$ there is a number n_0 such that for each $n \in \mathbb{N}$, $n > n_0$,

$$N(\frac{1}{n}, \frac{-1}{n}) = b(\frac{2}{n}) = \frac{5n-1}{5n} > \delta \text{ and simultaneously } |N(5, \frac{1}{n}) - N(5, \frac{-1}{n})| = \frac{1}{2} + \frac{1}{10n} > \frac{1}{2}.$$

If X is a nonempty set, N a nearness on X, $x_0 \in X$ and $\epsilon < 1$, let us denote:

$$O_\epsilon(x_0) = \{x \in X : N(x, x_0) > \epsilon\}.$$

Now denote by $\mathcal{B}$ the system of all $O_\epsilon(x)$, for all $x \in X$ and $\epsilon < 1$.

Theorem 4. *The system $\mathcal{B}$ creates a basis of a topology on X.*

Proof. It is sufficient to show, that if B_1 and B_2 are two sets from the system $\mathcal{B}$ and if $x_0 \in B_1 \cap B_2$, then there exists $B_0 \in \mathcal{B}$ such that $x_0 \in B_0 \subset B_1 \cap B_2$.

Let $B_1 = O_{\epsilon_1}(x_1)$, $B_2 = O_{\epsilon_2}(x_2)$ and $x_0 \in B_1 \cap B_2$. We will show that there exists $\delta_0 < 1$ such that $B_0 = O_{\delta_0}(x_0) \subset B_1 \cap B_2$:

Let $\epsilon_0 \in (0, \min(N(x_1, x_0) - \epsilon_1, N(x_2, x_0) - \epsilon_2))$ be arbitrary, but fixed.

From the property (N3) it follows, that there exists a number $\delta_0 < 1$ such that for each $x \in X$ and $i = 1, 2$

$$N(x, x_0) > \delta_0 \Longrightarrow |N(x, x_i) - N(x_0, x_i)| < \epsilon_0.$$

Thus if $x \in O_{\delta_0}(x_0)$ then $N(x, x_i) > N(x_0, x_i) - \epsilon_0 > \epsilon_i$ for $i = 1, 2$, so that $x \in O_{\epsilon_1}(x_1) \cap O_{\epsilon_2}(x_2)$.

From now on let $\mathcal{T}$ denote the topology created on the set X by the basis $\mathcal{B}$ and let $(X, \mathcal{T})$ denote the corresponding topological space.

Example 4. Let $X = \mathbb{R}$ and let N be a trivial nearness on X:

$$N(x,y) = \begin{cases} 1, & \text{for } x = y, \\ 0, & \text{for } x \neq y. \end{cases}$$

As is easy to check, N satisfies the properties (N1), (N2) and (N3). The basis $\mathcal{B}$, derived from the nearness N consists of all one-point sets, therefore the created topology $\mathcal{T}$ is discrete.

It can be simply verified, that any nearness in $\mathbb{R}$ of the type (b) establishes a basis $\mathcal{B}$, containing all one-point sets, thus in this case the topology $\mathcal{T}$ is always discrete.

Similarly , it can be easily proved, that despite for nearnesses in $\mathbb{R}$ of the type (a) we can obtain different bases, depending on the concrete nearness-generating functions, the topology $\mathcal{T}$ created by those bases is always the same - the standard topology of real axis.

It is possible to define and to investigate N-convergence, N-continuity and relevant notions in a universe X analogicaly as in the real case. The study in this direction will appear in the forthcoming article.

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