FUZZY LENGTH OF CURVES

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Abstract. The aim of this paper is to introduce the "fuzzy length" of curves. The concept is based on the notion of "fuzzy nearness relation", which is put into a metric space.

The formula which allows us to find the length of a given curve (in the Euclidean metric space) is well known. However, to use it we need precise information about the curve. And this is not always the case. Our long-term intension is to "measure" (in a vague sense) the length of a curve if this is given in a "fuzzy" sense. The first step will be the introducing a fuzzy length.

Our considerations will take place in a metric space $(\mathbb{X}, \rho)$. We assume that for each real $r > 0$ there are $x_1, x_2 \in \mathbb{X}$ such that $\rho(x_1, x_2) = r$. The crutial notion will be that of fuzzy nearness, introduced in [K] and further developed in [J] and in [D1,D2].

$\mathcal{M} : \mathbb{X} \times \mathbb{X} \to [0;1]$ is said to be the **relation of fuzzy nearness metrizable by** ρ iff there exists a continuous nonincreasing function $f : [0;\infty] \to [0,1]$ such that $f(0) = 1$ and $f(\infty) = 0$ and there holds

$$x\mathcal{M}y = f(\rho(x,y)).$$

Let us give some examples of such fuzzy nearness relations $\mathcal{M}$.

Examples.

(E1)
$$x\mathcal{M}y = \exp(-\rho(x,y))$$

(E2)
$$x\mathcal{M}y = \begin{cases} 1 - \rho(x,y) & \text{iff } \rho(x,y) \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(E3)
$$x\mathcal{M}y = \begin{cases} 1 & \text{iff } \rho(x,y) \le 0.5 \\ 2 - 2\rho(x,y) & \text{iff } 0.5 < \rho(x,y) \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$
(E4) \qquad x\mathcal{M}y = \begin{cases} 1 & \text{iff } \rho(x,y) \le 0.5 \\ 2 - 2\rho(x,y) & \text{iff } 0.5 < \rho(x,y) \le 0.75 \\ 0.5 & \text{iff } 0.75 < \rho(x,y) \le 1 \\ 1 - 0.5\rho(x,y) & \text{iff } 1 < \rho(x,y) \le 2 \\ 0 & \text{otherwise} \end{cases}
$$

As the main goal will be estimating a fuzzy length in an Euclidean space, we can assume, that there are no points in $\mathbb{X}$ isolated with respect to the nearness, that means for any pair of points $x, y \in \mathbb{X}$ and any $\alpha \in]0;1[$ there is a finite sequence of points $x = x_0, x_1, \ldots, x_n = y \in \mathbb{X}$ such that $x_k \mathcal{M} x_{k+1} \ge \alpha$.

Fuzzy distance of points. How to count distances (or, better, "distances") in the announced fuzzy sence? Let us fix a metric, ρ, and a fuzzy nearness relation, $\mathcal{M}$, metrizable by ρ. Up to the end of this article ρ and $\mathcal{M}$ will always mean the just mentioned couple. To each $\alpha \in]0;1]$ we can relate a unique number

$$
\rho_\alpha = \sup\{r \in \mathbb{R}; (\exists x, y \in \mathbb{X})(\rho(x,y) = r \& x\mathcal{M}y \ge \alpha)\}.
$$

We define the α **distance** ($\alpha \in]0;1[$) of two points $x, y \in \mathbb{X}$ by the folowing

$$
\tilde{\rho}_\alpha(x,y) = [n \cdot \rho_\alpha; (n+1) \cdot \rho_\alpha],
$$

where

$$
n = \begin{cases} 0 & \text{iff } x\mathcal{M}y \ge \alpha \\ \min\{k \ge 1; (x_0 = x \ \& \ x_0\mathcal{M}x_1 \ge \alpha \ \ldots \ x_{k-1}\mathcal{M}x_k \ge \alpha \ \& \ x_k\mathcal{M}y \ge \alpha\} \\ & \text{otherwise.} \end{cases}
$$

The **fuzzy distance** of the points x, y is given by

$$
\tilde{\rho}(x,y) = \lim_{\alpha \to 1_+} \tilde{\rho}_\alpha(x,y).
$$

This limit has to be taken with respect to the Hausdorff metric in the real line. As the sets $\tilde{\rho}_\alpha(x,y)$ are closed bounded intervals, their limit is again an interval (possibly collapsed to a singleton), whose left and right endpoints are the limits of the right and left endpoints in these intervals.

Lemma 1. *If $\rho_1 = 0$, then $\tilde{\rho}(x,y) = \rho(x,y)$. Otherwise $\tilde{\rho}(x,y)$ is an interval of the length ρ_1 containing $\rho(x,y)$.*

Comment. Due to Lemma 1 in case of fuzzy nearness relations (E1,E2) the fuzzy distance of points x, y is a precise number, which is identical with their distance $\rho(x,y)$. In case of fuzzy nearness relations (E3,E4) the fuzzy distance of x, y is an interval of the length ρ_1, i.e. in both cases of the length 0.5.

Fuzzy length of curves. In this section, for the simplicity, we will assume to work in an n-dimensional vector space $\mathbb{X}^n$, endowed with the metric, ρ, with its domain $\mathbb{X}^n \times \mathbb{X}^n$, and the fuzzy nearness relation, $\mathcal{M} : \mathbb{X} \times \mathbb{X} \to [0,1]$, similarly as it was in the above paragraph. Further, we will have our curve in question, $\mathfrak{C}$, given by a vector function $\mathfrak{C} = \{\varphi_i\}_{i=1}^n$, where $\varphi_i : \mathbb{X} \to \mathbb{X}$ are continuous functions. $\mathfrak{C}(t_1, t_2)$ will denote that $t_1 \in \mathbb{X}$ and $t_2 \in \mathbb{X}$ are the starting and ending points, respectively, of the parametrization of the curve $\mathfrak{C}$. $\mathfrak{C}(t)$ will denote the point $\mathfrak{C}(t) = \{\varphi_i(t)\}_{i=1}^n$.

The α**-length of the curve** $\mathfrak{C}(t_1, t_2)$ is defined by

$$\tilde{\rho}_{\mathfrak{C}(t_1,t_2),\alpha} = \begin{cases} [0, \rho_\alpha] & \text{iff for all } t \in [t_1; t_2]\ \mathfrak{C}(t)\mathrm{M}\mathfrak{C}(t_1) \geq \alpha \\ [k \cdot \rho_\alpha; (k+1) \cdot \rho_\alpha] & \text{otherwise,} \end{cases}$$

where

$$k = \min\{j \geq 1; (s_0 = t_1, s_{j+1} = t_2) \& (\forall i = 1, \ldots, j)(\rho(\mathfrak{C}(s_{i-1}), \mathfrak{C}(s_i)) \leq \rho_\alpha) \& \\ (\forall i = 0, 1, \ldots, j)(\forall t \in [s_i; s_{i+1}])(\rho(\mathfrak{C}(t), \mathfrak{C}(s_i)) \leq \rho_\alpha)\}.$$

Theorem 1. *The α-length of a curve $\mathfrak{C}(t_1, t_2)$ is independent of its parametrization.*

The curve $\mathfrak{C}(t_1, t_2)$ will be called **fuzzy rectificable**, iff for all $\alpha \in]0,1[$ its α-length is finite.

We say that the curve $\mathfrak{C}(t_1, t_2)$ **is of finite fuzzy length,** given by

$$\tilde{\rho}_{\mathfrak{C}(t_1,t_2)} = \lim_{\alpha \to 1_+} \tilde{\rho}_{\mathfrak{C}(t_1,t_2),\alpha}$$

if the resulting interval has a finite upper bound.

Example. Let us consider the Euclidean metric space E^2 and let us have the following curve

$$\varphi_1(t) = t,\ \varphi_2(t) = \frac{\sin t}{t}, \quad t \in [-1; 1].$$

Then regardless of fuzzy nerness relation this curve is fuzzy rectificable and in case the fuzzy nearness relation has $\rho_1 \neq 0$ (relations (E3,E4)), it is of finite fuzzy length.

Theorem 2. *If for the fuzzy nearness relation M $\rho_1 = 0$ holds, then there are fuzzy rectificable curves with no fuzzy length in our metric space.*

References

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