ON THE ENTROPY OF T - FUZZY DYNAMICAL SYSTEMS

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Recently A. Ban ([1]) introduced the entropy of a dynamical system with respect to a given T - norm. In this contribution we present some variants of the Kolmogorov - Sinai theorem on connections between the entropy of a dynamical system and the entropy of its generator.

Assumptions

As it is very well know a t - norm T is a mapping $T : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$ which is commutative, associative, monotone and such that $T(x,1) = x$ for any x. The associated t - conorm S is defined by the formula

$$S(x,y) = 1 - T(1-x, 1-y)\,.$$

We shall work with Frank's t - norms ([4]). A t - norm T is Frank if and only if

$$T(x,y) + S(x,y) = x + y$$

for any $x, y \in \langle 0,1\rangle$.

Further we assume that a σ - algebra $\mathbf{S}$ of subsets of a set X is given and we consider the family F all functions $f : X \to \langle 0,1\rangle$ measurable with respect to $\mathbf{S}$. A probability T - measure is given, i.e. a mapping $m : F \to \langle 0,1\rangle$ satisfying the follwing conditions:

(i) $m(1_X) = 1$;

(ii) $m(T(f,g)) + m(S(f,g)) = m(f) + m(g)$ for any $f, g \in F$;

(iii) If $f_n \nearrow f$, then $m(f_n) \nearrow m(f)$.

By an excellent Klement theorem ([2], Corollary 5.9) to any Frank t - norm T and any probability T - measure m there exists a probability measure P on the generating σ - algebra $\mathbf{S}$ and a Markov kernel $K : X \times B(R) \to \langle 0,1\rangle$ satisfying the following conditions:

(i) $K(\cdot, A) : X \to \langle 0,1\rangle$ is an $\mathbf{S}$ - measurable function for any $A \in \mathbf{S}$;

(ii) $K(x,\cdot) : B(R) \to \langle 0,1\rangle$ is a probability for any $x \in X$;

(iii) $m(f) = \int_X K(x, \langle 0, f(x)\rangle) dP(x)$ for any $f \in F$.

Entropy of a fuzzy T - dynamical system

The basic notion of the Ban theory is the notion of a T - fuzzy partition, which is a family $D = \{f_1, \cdots, f_k\}$ of functions $f_i : X \to \langle 0,1\rangle$ such that

(i) $\mathrm{S}_{i=1}^{n} f_i = 1_x$;

(ii) $(\mathrm{S}_{i\neq k} f_i)\; T\, f_k = 0_x$ for any k.

If $D = \{f_1, \cdots, f_k\}$, $B = \{g_1, \cdots, g_m\}$ are two T - fuzzy partitions, then

$$D \vee B = \{f_i \cdot g_j\, ; \quad i = 1, \cdots, k, \quad j = 1, \cdots, m\},$$

where $f_i \cdot g_j$ is the usual product of two real functions. The entropy of a T - fuzzy partition D is defined by

$$H(D) = \sum_{i=1}^{k} \varphi(fi)\,,$$

where $\varphi(x) = -x \cdot \log x$, if $x > 0$, $\varphi(0) = 0$. For defining a dynamical system we need a transformation $U : X \to X$ such that

$$f \in F \Rightarrow f \circ U \in F \quad \text{and} \quad m(f \circ U) = m(f)\,.$$

The entropy $h(D, U)$ of a T - fuzzy partition D with respect to U is the limit

$$h(D,U) = \lim_{n\to\infty} \frac{1}{n} \cdot H\left(\bigvee_{i=0}^{n-1} U^{-i}(D)\right)\,,$$

where $U^{-i}(D) = \{f_1 \circ U^i, \cdots, f_k \circ U^i\}$. The entropy $h(U)$ of the dynamical system (X, F, m, U) is defined by the formula

$$h(U) = \sup\{h(D,U)\,; \quad D \text{ is a } T \text{ - fuzzy partition}\}.$$

Theorem

Let T be Frank t - norm, F be a T - tribe (generated by a σ - algebra $\mathbf{S}$), m be a probability T - measure on F (generated by a probability measure P and a Markov kernel K), $B = \{A_1, \cdots, A_k\}$ be a crisp partition of X generating $\mathbf{S}$ (i.e. $\sigma\left(\bigcup_{i=0}^{\infty} U^{-i}(B)\right) = \mathbf{S}$).

Then for every fuzzy T - partition $D = \{f_1, \cdots, f_m\}$

$$h(A, U) \leq h(B, U) + \sum_i \int_X h_i dP\,,$$

where $h_i : X \to R$ is defined by the formula $h_i(x) = K(x, \langle 0, f_i(x)))$.

Proof. [6], Theorem 1.

Corollaries

Let all previous assumptions be satisfied. We shall define the entropy of a T - fuzzy partition by a different way ([3]):

$$\widehat{H}(D) = \sum_i \varphi(m(f_i)) - \sum_i \int_X \varphi(K(x, \langle 0, f_i(x)))dP(x)\,.$$

The entropy of the dynamical system is then defined as follows:

$$\widehat{h}(D, U) = \lim_{n\to\infty} \frac{1}{n} \cdot \widehat{H}\left(\bigvee_{i=0}^{n-1} U^{-i}(D)\right)\,,$$

$$\widehat{h}(U) = \sup\{\widehat{h}(D, U)\,; \quad D \text{ is } T \text{ - fuzzy partition}\}.$$

Corollary 1. *If B is a crisp generator, then*

$$\widehat{h}(U) = \widehat{h}(B, U) = h(B, U)\,.$$

Finally we examine a special case of the Lukasiewicz t - norm T_L (T_∞ in the Frank notation):

$$T_L(x, y) = max(x + y - 1, 0)\,.$$

In this case the T_L - fuzzy partition is a finite set $D = \{f_1, \cdots, f_k\} \subset F$ such that $\sum_{i=1}^{k} f_i = 1_X$. Moreover, the Markov kernel K has the form $K(x, \langle 0, f(x))) = f(x)$ ([2], theorem , [5], theorem), hence

$$m(f) = \int_X f dP\,.$$

Corollary 2. *([5]). Let F be the set of all fuzzy subsets of a set X measurable with respect to a σ - algebra $\mathbf{S}$. If B is a measurable crisp partition generating $\mathbf{S}$, then*

$$h(D,U) \leq h(B,U) + \sum_i \int_X \varphi(f_i)dP$$

for every T_L - fuzzy partition $D = \{f_1, \cdots, f_k\}$.

Corollary 3. *([5]). Let F be the set of all fuzzy subsets of a set X measurable with respect to a σ - algebra $\mathbf{S}$. If B is a measurable crisp partition generating $\mathbf{S}$, then*

$$\widehat{h}(U) = h(B,U) = \widehat{h}(B,U).$$

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