Wd—Fuzzy Implication Algebras

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Abstract: This paper is about a kind of abstract algebra which is different from FI algebra and HFI algebra. Wd — FI algebar is abbreviation for it, we show the internal relations between it and regular FI algebra and HFI algebra, and discussed some properties of it.

Keywords: Wd—Fuzzy implication; Regular FI algebra; HFI algebras

1. Introduction

Wu[1] defined Fuzzy implication algebras and obtained some elementary properties. In this paper we define Wd—Fuzzy implication algebra and prove it's properties.

2. Wd—FI algebra

We start this section define Wd — FI algebra, then give some properties of Wd—FI algebra.

Definition 2. 1[1] An algebraic system $(X, \to, 0)$ of type $(2,0)$ is called a FI—algebra if it satisfies the following conditions:

(I_1) $x \to (y \to z) = y \to (x \to z)$

(I_2) $(x \to y) \to [(y \to z) \to (x \to z))] = 1$

(I_3) $x \to x = 1$

(I_4) $x \to y = y \to x = 1$ impies $x = y$

(I_5) $0\to x=1$ where $1=0\to 0$

for all $x,y,z\in X$.

Definition2. 2[1] An algebraic system $(X,\to,0)$ of type (2,0) is called HFI algebras if it satisfies the following conditions:

(H_1) $x\to(y\to x)=1$

(H_2) $[x\to(y\to z)]\to[(x\to y)\to(x\to z)]=1$

(H_3) if $1\to x=1$, then $x=1$

(H_4) if $x\to y=y\to x=1$, then $x=y$

(H_5) $0\to x=1$ where $1=0\to 0$

Definition 2. 3 An algebraic system $(X,\to,0)$ of type (2,0) is called Wd—FI algebras if it satisofies the following conditions:

(1) $x\to(y\to z)=y\to(x\to z)$

(2) $(x\to y)\to z=(z\to y)\to x$

(3) $x\to x=1$

(4) $x\to y=y\to x=1$ implies $x=y$

(5) $0\to x=1$ where $1=0\to 0$ for all $x,y,z\in X$.

Example 1. Let $X=\{0\}$ define $0\to 0=0$ (whene $1=0$) then $(X,\to,0)$ is a Wd—FI algbras.

Example 2. Let $X=[0,1]$, define $x\to y=1$, for all $x,y\in X$, then $(X,\to,0)$ is a Wd—FI algebras.

Example 3. Let $X=\{0,a,1\}$, $\to$: $X\to X$ be defined by

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

then $(X,\to,0)$ is a FI algebra but not Wd—FI algebra. Because of $(1\to a)\to 0=a\to 0=0$; $(0\to a)\to 1=1\to 1=1$, so (2) is false.

In order prove that every Wd—FI algebra are FI algebra, We first proved some Lemmas in first section. $(X, \to, 0)$ are Wd—FI algebra in the following Lemma.

Lemma 1. For any $x \in X$, we have $x \to 1 = 1, 1 \to x = x$.

Proof. By $x \to 1 = x \to (0 \to x) = 0 \to (x \to x) = 0 \to 1 = 1$ so, $x \to 1 = 1$.

$$(1 \to x) \to x = (x \to x) \to 1 = 1 \to 1 = 1$$

$$x \to (1 \to x) = (x \to x) \to 1 = 1 \to 1 = 1$$

by(4) we have $1 \to x = x$.

Lemma 2. For any $x, y, z \in X$, if $x \to y = 1$, $y \to z = 1$ then $x \to z = 1$

Proof. By Lemma 1 $1 \to x = x$, we have

$$x \to z = 1 \to (x \to z) = (x \to y) \to (x \to z)$$

$$= [(x \to z) \to y] \to x = [(y \to z) \to x] \to x = x \to x = 1,$$

hence $x \to z = 1$

Let $(X, \to, 0)$ be a Wd—FI algebra, we define the relation $\leqslant$: $x \leqslant y$ iff $x \to y = 1$, $x, y \in X$, then $(X, \leqslant)$ is a partial order sets.

Lemma 3. For any $x, y, z \in X$, if $x \leqslant y$,

then $z \to x \leqslant z \to y$, $y \to z \leqslant x \to z$

Proof. By (1),(2), $(z \to x) \to (z \to y) = z \to [(z \to x) \to y]$

$$= z \to [(y \to x) \to z] = (y \to x) \to (z \to z) = (y \to x) \to 1$$

$$= (1 \to x) \to y = x \to y = 1, \text{ hence } z \to x \leqslant z \to y.$$

Since $(y \to z) \to (x \to z) = x \to [(y \to z) \to z]$

$= x \to [(z \to z) \to y] = x \to (1 \to y) = x \to y = 1$, hence $y \to z \leqslant x \to z$.

Lemma 4. For any x, y, $z \in X$, we have that

$$(x \to y) \to [(y \to z) \to (x \to z)] = 1$$

Proof. By(1), (2), we have $(x \to y) \to [(y \to z) \to (x \to z)]$

$$= (y \to z) \to [((y \to z) \to x) \to x]$$

$$= (y \to z) \to [(x \to z) \to (y \to z)] = (y \to z) \to [1 \to (y \to z)]$$

$$= (y \to z) \to (y \to z)] = 1.$$

By above Lemmas and definition 2.3, we have:

Theorem 2. 1 Any Wd—FI algebra is a FI algebra.

3. Wd—FI algebra and regular FI algebra

Definition 3. 1 [1] Let $(X, \to, 0)$ be a FI algebra, define a unary operation C: $C(x)=x\to 0$, for any $x\in X$, $C(x)$ is called pseudo—complement of x.

Definition 3. 2 [1] Let $(X,\to,0)$ be a FI algebra, $(X, \to,0)$ is a rebular FI algebra iff for any $x\in X$, we have that $CC(x)=x$.

Theorem 3. 1 Any Wd — FI algebra is a regular FI algebra, in reverse is not holds

Proof. By theorem 2. 1 and definition 3. 2, we only proved that for any $x\in X$ implies $CC(x)=x$. In fact, $CC(x)=(x\to 0)\to 0=(0\to 0)\to x =1\to x=x$, the proof is completed.

Example. Let $X=[0,1]$, define $x\to y=\min(1,1-x+y)$, then $(X,\to,0)$ is a regular FI algebra, but it is not Wd—FI algebra.

In fact, Let $x=0.1$, $y=0.2$, $z=0.3$, because

$$(0.1\to 0.2)\to 0.3=\min(1,1-0.1+0.2)\to 0.3=1\to 0.3=0.3$$

$$(0.3\to 0.2)\to 0.1=\min(1,1-0.3+0.2)\to 0.1=0.9\to 0.1$$

$$=\min(1,1-0.9+0.1)=0.2$$

hence $(0.1\to 0.2)\to 0.3=0.3\neq 0.2=(0.3\to 0.2)\to 0.1$

so $(x\to y)\to z=(z\to y)\to x$ is not hold.

It follows from theorem 3. 1 we have that:

Wd—FI algebra is proper subclass of regular FI algebra.

Theorem 3. 2 Let $(X,\to,0)$ be a Wd—FI algebra, We have that,

(i) $y\to z=(x\to y)\to(x\to z)$

(ii) $[(y\to x)\to(z\to x)]\to(z\to y)=1$

(iii) $[x\to(x\to y)]\to y=x$

(iv) $(x\to y)\to[(y\to x)\to y]=y$

(v) $(x\to y)\to y=x$

(vi) $[x\to(y\to z)]\to[(x\to y)\to(x\to z)]=x$

(vii) $[(x\to y)\to z]\to[(x\to z)\to y]=(y\to z)\to(z\to y)$

(viii) $x\to[y\to(y\to x)]=1$

(IX) $(x\to z)\to(y\to z)=y\to x$

proof. (i) Because $(x\to y)\to(x\to z)=[(x\to z)\to y]\to x$

$=[(y\to z)\to x]\to x=(x\to x)\to(y\to z)=1\to(y\to z)$

$=y\to z$, hence $y\to z=(x\to y)\to(x\to z)$

(ii) $[(y\to x)\to(z\to x)]\to(z\to y)$

$=[(z\to y)\to(z\to x)]\to(y\to x)$

$=(y\to x)\to(y\to x)=1$

(iii) $[x\to(x\to y)]\to y=[y\to(x\to y)]\to x$

$=[x\to(y\to y)]\to x=(x\to 1)\to x=1\to x=x$

(iv) $(x\to y)\to[(y\to x)\to y]=(y\to x)\to[(x\to y)\to y]$

$=(y\to x)\to[(y\to y)\to x]=(y\to x)\to x$

$=(x\to x)\to y=1\to y=y$

(v) $(x\to y)\to y=(y\to y)\to x=1\to x=x.$

(vi) $[x\to(y\to z)]\to[(x\to y)\to(x\to z)]$

$=[x\to(y\to z)]\to[(x\to y)\to(x\to z)]$

$=[x\to(y\to z)]\to(y\to z)$

$=[(y\to z)\to(y\to z)]\to x=1\to x=x$

(vii) $[(x\to y)\to z]\to[(x\to z)\to y]$

$=[(z\to y)\to x]\to[(y\to z)\to x]$

$=(y\to z)\to\{[(z\to y)\to x]\to x\}$

$=(y\to z)\to[(x\to x)\to(z\to y)]$

$=(y\to z)\to(1\to(z\to y))$

$=(y\to z)\to(z\to y)$

(viii) $x\to[y\to(y\to x)]=y\to[x\to(y\to x)]$

$=y\to[y\to(X\to x)]=y\to(y\to 1)=y\to 1=1$

(IX) $(x\to z)\to(y\to z)=y\to[(x\to z)\to z]$
$=y\to[(x\to z)\to z]=y\to[(z\to z)\to x]$
$=y\to(1\to x)=y\to x.$

4. **Wd—FI algebra and HFI algebra.**

By theorem 3. 2 and definition 2. 2 we have that:

Theorem 4. 1 Wd—FI algebra may not be HFI algebra.

Theorem 4. 2 Let $(X,\to,0)$ be a Wd—FI algebra ,if any $x,y,z\in X$, We have $x\to(y\to z)=(x\to y)\to(x\to z)$ holds, then Wd—FI certain be a HFI algebra.

Proof. Let $(X,\to,0)$ be a Wd—FI algebra, it is satisfies the condition (H_4), (H_5) of defintion 2. 2, and $x\to(y\to z)=y\to(x\to z)$, let $z=x$, we have $x\to(y\to x)=y\to 1=1$, hence (H_1) hold; according to it following the condition $x\to(y\to z)=(x\to y)\to(x\to z)$, we have $[z\to(y\to z)]\to[(x\to y)\to(x\to z)]=1$. so (H_2) hold; because $1\to x=x$ and $1\to x=1$, we have $x=1$, so (H_3) hold. Hence $(X,\to,0)$ be a HFI algebra.

The following theorem can be easily proved by reference [1]

Theonem 4. 3 Let $(X,\to,0)$ be a HFI algebra, if $(x\to y)\to z=(z\to y)\to x$, for any $x,y,z\in X$, then $(X,\to,0)$ sure be a Wd—FI algebra.

Reference

[1] Wu Wang Ming, Fuzzy implication algebra, Fuzzy System and Mathematics (in China) (1) 1990: 58~63.

[2] Deng Fang´an Some results of Fuzzy implication algebras, Hanzhong tecahers college, (2) 1992:16~20.