CHEBYSHEV'S FORM OF THE LAW OF LARGE NUMBERS FOR FUZZY NUMBERS

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In this paper Chebyshev's law of large numbers for fuzzy numbers in the framework of the possibility theory is representing. Results are formulating in terms of the additive generator of a triangular norm.
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1. **Introduction.** In thise paper Chebyshev's fuzzy law of large numbers in the framework of theory possibility is formulate. Resently [2] was shown that for fuzzy numbers of symmetric triangular form X_1, X_2,... the law of large numbers is obeys. In [3] Dombi's operator used and the fuzzy law of large numbers for more general environment is shown as well. Then t-norm representation theorem of Ling is used as a basic tool and the results are presented in terms of the additive generator of a triangular norm [4]. Here we developed that research.

Note, that membership function a symmetric triangular fuzzy number $X_i= (m_i,d_i)$, is defined as

$\mu_{X_i}(x) = \max(0, 1-|x-m_i|/d_i)$, d_i- is its width; m_i- is its modal value ($d_i>0$, $-\infty < m_i < \infty$). They are an L-R type fuzzy numbers by Dubois & Prade [1], when $L(x) = R(x) = \max(0, 1 - |x|)$:

$$\mu_{X_i}(x) = \begin{cases} L((m_i-x)/d_i), & \text{for } x \leqslant m_i \\ R((x-m_i)/d_i), & \text{for } x \geqslant m_i \end{cases}$$

Now, the grade of the possibilty of the statement: " [a,b] contains the value of X" is defined as [2]

$$\text{Pos}(a \leqslant X \leqslant b) = \sup_{a<x<b} \mu_X(x);$$

And necessity is defined as

$$\text{Nes}(a \leqslant X \leqslant b) = 1 - \text{Pos}(X<a, X>b).$$

Now [1], function T: [0;1] x [0;1] --> [0;1] is t-norm, if T is commutative, associative, non-decreasing and $T(0,1) = 0$, $T(1,1) = 1$. A t-norm will be called Archimedean if T is continuous and $T(u,u) < u$; $0 < u < 1$.

T-sum of two fuzzy numbers is denoted as $S_T = (X_1+X_2)_T$ and its membership function is defined as

$$\mu_{S_T}(z) = \sup_{x+y=z} T(\mu_{X_1}(x), \mu_{X_2}(y))$$

Obviously, that for a tasks of applicable character it is interesting to study the behavior of the T-sum of fuzzy nambers $S_n = ((w_1X_1+w_2X_2+\ldots+w_nX_n))_T$ when $n \to \infty$ and $w_1+w_2+\ldots+w_n=1$, $w_i \geqslant 0$, $i=1,2,\ldots,n$.

2. **Results.**

We consider one of the case L-R type fuzzy numbers by Dubois & Prade when membership function of fuzzy number X_i is $\mu_{X_i}(x) = L(|x-m_i|/d)$. Here L is a decreasing function on $[0;\infty)$, and $L(0) = 1$, $L(x) = L(-x)$.

In this the case we'll speak that the fuzzy number X_i belongs to the class L_L, $X_i \in L_L$.

Theorem 1. If T is Archimedian t-norm, $X_i \in L_L$, $w_1+\ldots+w_n=1$ then for any $\varepsilon>0$

$$\text{Nes}\left[N_n-\varepsilon \leqslant \left(w_1X_1+w_2X_2+\ldots+w_nX_n \right)_T \leqslant N_n+\varepsilon\right] =$$

$$= f^{-1}\left(\min\ (f(0),\ n\cdot f(L(\varepsilon/d))\right), \quad N_n=w_1m_1+w_2m_2+\ldots+w_nm_n,$$

f - is an additive generator of a triangular norm T,

f^{-1} - is its inverse.

Theorem 2. If T is Archimedian t-norm, $X_i \in L_L$, $w_1+\ldots+w_n=1$ then for any $\varepsilon>0$

$$\text{Nes}\left[N_n-\varepsilon \leqslant \left(w_1X_1+w_2X_2+\ldots+w_nX_n \right)_T \leqslant N_n+\varepsilon\right] \geqslant 1 - L(\varepsilon/d)$$

Theorem 3. If T is Archimedian t-norm, $X_i = (m_i, d)$, then for any $\varepsilon>0$

$$\text{Nes}\left[M_n-\varepsilon \leqslant \left(\frac{X_1+X_2+\ldots+X_n}{n} \right)_T \leqslant M_n+\varepsilon\right] = 1-f^{-1}\left(\min\ (f(0),\ n\cdot f(1-\varepsilon/d))\right)$$

$M_n = (m_1+m_2+\ldots+m_n)/n$,

From theorem 3 we can get some useful corollaries.

Corollary 1. For the environment of theorem 3, when $f(0) = 1$ or $f(0) = \infty$, we have

$$\lim_{n \to \infty} \text{Nes}\left[M_n-\varepsilon \leqslant \left(\frac{X_1+X_2+\ldots+X_n}{n} \right)_T \leqslant M_n+\varepsilon\right] = 1$$

Corollary 2. Let $T(u,v) = u\cdot v$, $X_i \in L_L$ and $L(x) = \max(0,\ 1 - x^2)$, then for $0 < \varepsilon < d$

$$\mathrm{Nes}\left[\frac{m_1+m_2+\ldots+m_n}{d\sqrt{n}} - \varepsilon \leqslant \left(\frac{X_1+X_2+\ldots+X_n}{d\sqrt{n}}\right)_T \leqslant \frac{m_1+m_2+\ldots+m_n}{d\sqrt{n}} + \varepsilon\right] =$$

$= 1 - \exp(-\varepsilon^2)$.

Particulary, if $\varepsilon = \sqrt{3}$, we have

$$\mathrm{Nes}\left[\frac{m_1+m_2+\ldots+m_n}{d\sqrt{n}} - \sqrt{3} \leqslant \left(\frac{X_1+X_2+\ldots+X_n}{d\sqrt{n}}\right)_T \leqslant \frac{m_1+m_2+\ldots+m_n}{d\sqrt{n}} + \sqrt{3}\right] \approx 0.95$$

3. **Proof of theorems.** We are begining from theorem 1. If T is Archimedian t-norm, $p+q=1$, $p > 0$, $S_T = (pX_1+qX_2)_T$ then out of the definition of T-sum and Ling's theorem [1], for a fixed $z=z^*$ we have:

$$\mu_{S_T}(z^*) = \sup_{px+qy=z} T(\mu_{X_1}(x),\ \mu_{X_2}(y)) = \sup_x T(\mu_{X_1}(x),\ \mu_{X_2}((z^*-px))/q) =$$

$$= \sup_x f^{-1}\left(\min\ (f(0),\ f(\mu_{X_1}(x)) + f(\mu_{X_2}((z^*-px))/q)\right)$$

Let m1<m2. We'll consider the proof taking into consideration only the left parts of membership functions for fuzzy numbers X_1, X_2, which have the following type:

$$\mu_{X_1}(x) = L\left(\frac{m_1-x}{d}\right), m_1-d \leqslant x \leqslant m_1;\quad \mu_{X_2}(y) = L\left(\frac{m_2-y}{d}\right),\ m_2-d \leqslant y \leqslant m_2;$$

$$\mu_{X_2}((z^*-px)/q) = L\left(\frac{qm_2-z^*+px}{qd}\right),\quad (z^*-qm_2)/p \leqslant x \leqslant (z^*-qm_2+qd)/p;$$

Taking into account that an additive generator $f: X \longrightarrow [0;1]$ is a continious and decreasing function with $f(1) = 0$, it's easy to see that $f(\mu_{X_1}(x))$ is a deacreasing while $f(\mu_{X_2}((z^*-px)/q)$ is an increasing function on the interval

$$\max((z^*-qm_2)/p;\ m_1-d) \leqslant x \leqslant \min((z^*-qm_2+qd)/p;\ m_1).$$

Then the minimum value of the sum: $f(\mu_{X_1}(x)) + f(\mu_{X_2}((z^*-px)/q)$ can be found as a solution of the following equation:

$$L\left(\frac{m_1-x}{d}\right) = \lambda = L\left(\frac{qm_2-z^*+px}{qd}\right).\ \text{It's equal to}\ 2\cdot f(L\left(\frac{pm_1+qm_2-z^*}{d}\right)).$$

This value reached by $x = z^*+qm_1-qm_2$.

The same holds true for the right parts of membership functions for fuzzy numbers X_1, X_2.

If me write $w_1X_1+w_2X_2+\ldots+w_nX_n =$
$= (W_1X_1+\ldots+W_{n-1}X_{n-1})(w_1+\ldots+w_{n-1})+w_nX_n$, $W_i= w_i/(w_1+\ldots+w_{n-1})$, then we returned to the case of two fuzzy numbers.

Now it is clear that for $S_n = ((w_1X_1+w_2X_2+\ldots+w_nX_n))_T$

$$\mu_{S_n}(z) = f^{-1}\left(\min\ (f(0),\ n\cdot f(L(|z-N_n|/d)\right) \quad \text{and} \quad Nes\ (|S_n-N_n| \leqslant \varepsilon) =$$

$$= 1-Pos\ (|S_n-N_n| > \varepsilon) = 1- \sup_{\substack{z \\ |z-N_n|>\varepsilon}} \mu_{S_n}(z) = 1 - \sup_{\substack{z \\ |z-M_n|>\varepsilon}} f^{-1}\left(\min\ (f(0),\right.$$

$$\left. n\cdot f(L(|z-N_n|/d)\right) = 1 - f^{-1}\left(\min\ (f(0),\ n\cdot f(L(\varepsilon/d)\right).$$

Which completes the proof of the theorem 1.

Proof theorem 2. First consider the case n=2. Then taking into account that $T(u,v) \leqslant \min(u,v)$, by the proof of theorem 1 it is not difficult to see that

$$\mu_{S_T}(z) = \sup_{px+qy=z} T(\mu_{X_1}(x),\ \mu_{X_2}(y)) \leqslant \sup_{px+qy=z} \min\ (\mu_{X_1}(x),\ \mu_{X_2}(y)) =$$

$$= L(|z-(pm_1+qm_2)|/d).$$

And hense for $S_n = ((w_1X_1+w_2X_2+\ldots+w_nX_n))_T$ we have

$$Nes\ (|S_n-N_n| \leqslant \varepsilon) = 1- \sup_{\substack{z \\ |z-N_n|>\varepsilon}} \mu_{S_n}(z) \geqslant 1 - L(\varepsilon/d).$$

Theorem 3 follows as a corollary of the proof of theorem 1 and we omited its proof.

Proof of the Corollary 1. Let $f(0) = \infty$. Then taking into account that the additive generator f is a continious and decreasing function, for any fixed ε^* we have

$$\lim_{n \to \infty} f^{-1}\left(n\cdot f(1-\varepsilon^*/d)\right) = 0.$$

If $f(0) = 1$, then $\lim_{n \to \infty} f^{-1}\left(\min\ (f(0),\ n\cdot f(1-\varepsilon^*/d))\right)=f^{-1}\left(f(0)\right)= 0.$

Proof Corollary 2 is obviously if we taking into acount that for this case the additive generator is $f(x) = -\ln(x)$.

4. **Examples.**

We'll consider some examples using our theorem 3.

1. As a triangular norm T we'll choose Yager's operator [1]:

$$T_Y(u,v) = 1 - \min\left(1,\ (1-u)^q + (1-v)^q)\right)^{1/q}, \quad 0 \leqslant q < \infty$$

its additive generator is $f(x) = (1-x)^q$, $f(0)=1$, $f^{-1}(y)=1-y^{1/q}$.

Using that we'll calculate the right part of our theorem 3 $f(1-\varepsilon/d) = \left(\varepsilon/d\right)^q$, $f^{-1}\left(\min\ (f(0),\ n\cdot(\varepsilon/d)^q\right) = \max(0, 1-n^{1/q}\cdot(\varepsilon/d))$.

Hence, the law of large numbers in this case works.

If we will consider a special case when $q = \infty$ then we will have $\mathrm{Nes}\ (M_n-\varepsilon \leqslant S_n \leqslant M_n+\varepsilon) = \varepsilon/d$. Hence the fuzzy law of large numbers does not work [2,3,4].

2. As a triangular norm T we'll choose Dombi's operator. Its additive generator is

$$f(x) = \left((1-x)/x\right)^p,\ p > 0;\ f(0) = \infty;\ f^{-1}(y) = 1/(1+y^{1/p})$$

When p=1 then we have Hamacher's operator with zero parameter.

Next $n\cdot f(1-\varepsilon/d) = n\cdot\{\ (\varepsilon/d)\ /\ (1-\varepsilon/d)\ \}^{1/p}$, $f^{-1}\left(n\cdot f(1-\varepsilon/d)\right) =$
$= (1-\varepsilon/d)/(1 + (n^{1/p}-1)\cdot\varepsilon/d)$.

Therefore $\lim\limits_{n\to\infty} \mathrm{Nes}\ (M_n-\varepsilon \leqslant S_n \leqslant M_n+\varepsilon) = 1$ and the fuzzy law of large numbers works.

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