# On Fuzzy Singular Languages

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## ABSTRACT

In this paper, we have introduced the notion of fuzzy left (right, bi-) singular languages, and researched their algebraic structures and properties. In section 2 two equivalent conditions of fuzzy left (right, bi-) singular languages (th2.1, th2.2) have been given, and the relationship between fuzzy prefix (suffix, bifix) codes and fuzzy left (right, bi-) singular words has been researched (th2.4, th2.5). In section 3 the fuzzy strongly singular languages have been discussed, and the conclusion that the class of all fuzzy strongly singular languages is a submonoid of F(X*) has been obtained (th3.1). In Section 4 we have introduced the concept of cancellative property to research fuzzy singular languages, Some depictions of fuzzy singular languages by Cancellative property have been given (th4.1, th4.3, th4.6).

Keywords: Fuzzy semigroup; Fuzzy language; Fuzzy code; Fuzzy left (right, bi-) singular language

## 0. INTRODUCTION

Since Zadeh [3] in 1969 introduced the concept of fuzzy language the theory has been developed by many researchers. In the study of fuzzy formal language, a lot of excellent results have been achieved by researchers. In particular, the study of the properties of fuzzy grammar, the rules of fuzzy syntaxes ,and the recognition ability of a fuzzy automaton extended the applicable area of fuzzy set theory and reduced the difference between formal language and natural language [1,2,3].In[4], fuzzy code is studied using the method of fuzzy algebraic structures. In this paper, we also use the method of fuzzy algebraic structures to describe formal languages such as fuzzy left (right, bi-) singular language and discuss their algebraic properties. Furthermore, we give some depictions of fuzzy singular languages by cancellative property.

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## 1. PRELIMINARIES

In the following text, we suppose that X is an alphabet with $1 \leqslant |X| < \infty$, and $X^+$ ($X^*$) is the semigroup (free monoid) generated from X with the operation of adjoin , F stands for "fuzzy ", and F(X) denotes the set of all fuzzy subsets of X, $A \in F(X^*)$ is called F-language on the free monoid $X^*$, and 1 is the identity of $X^*$.

**Definition1.1[4].** Let $A \in F(X^+)$, $B \in F(X^+)$, for any $x \in X^+$, (A-B)(x)=A(x) if B(x)=0; (A-B)(x)=0 if B(x)>0.

**Definition 1.2.** Let A, $B \in F(X^*)$, for any $x \in X^*$, (AB)(x)=sup {min(A(y),B(z)) $y,z \in X^*$ yz=x}.

**Definition 1.3[4]**. A nonempty F-language $\phi \neq A \in F(X^+)$ is a F-prefix (suffix) code if $A \cap AX^+ = \phi$ ($A \cap X^+A = \phi$).

A is F-bifix code if A is F-prefix code and F-suffix code.

**Definition1.4[5,6].** Let $A \subseteq X^+$, let

$$l(A) = \{g \in A | gx \notin A \text{ for all } x \in X^* \text{ and } g = yz,\ z \in X^- \text{ implies } \quad y \notin A\}$$

$$r(A) = \{g \in A | xg \notin A \text{ for all } x \in X^- \text{ and } g = zy,\ z \in X^+ \text{ implies } y \notin A\};$$ and

$b(A)=L(A) \cap r(A)$. Every $g \in L(A)$ (r(A), b(A)) is called a left (right ,bi-) singular word in A. If $L(A) \neq \phi$ ($r(A) \neq \phi$, $b(A) \neq \phi$), tHen we call A an ordinary left (right, bi-) singular language. Pu and Liu gave the definition of a fuzzy point [7].

**Definition1.5.** Let X be an alphabet, $\forall x \in X$, $\lambda \in (0,1]$, a map $x_\lambda$: X→[0,1] such that $\forall y \in X$, $x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$, we call $x_\lambda$ a fuzzy point on X.

**Definition1.6.** Let $A \in F(X^*)$, we say that a fuzzy point $x_\lambda$ belongs to fuzzy set A if $\lambda \leqslant A(x)$. We denote it by $x_\lambda \in A$, and we have.

**Proposition1.7.** Let $A \in F(X^*)$, $A \neq \phi$, then $A = \bigvee_{x_\lambda \in A} x_\lambda$.

**Definition1.8.** S is called a strongly prefix (suffix) submonoid of monoid M if $M^{-1}S \subseteq S$ ($SM^{-1} \subseteq S$), where $M^{-1}S = \{x \in M |$ there exists $a \in M$ such that $ax \in S\}$ ($SM^{-1} = \{x \in M|$ there exists $a \in M$ such that $xa \in S$ }).

## 2. FUZZY SINGULAR LANGUAGE

**Definition2.1.** Let $\phi \neq A \in F(X^+)$, for any $g \in X^+$, let

$$L(A)(g) = \begin{cases} (A - AX^+)(g) & \text{if } A(gx) = 0 \text{ for all } x \in X^+ \\ 0 & \text{otherwise} \end{cases}$$

$$R(A)(g) = \begin{cases} (A - X^+A)(g) & \text{if } A(xg) = 0 \text{ for all } x \in X^+ \\ 0 & \text{otherwise} \end{cases}$$ and $B(A)=L(A) \cap R(A)$

Every $g_\lambda \in L(A)$ ($g_\lambda \in R(A)$, $g_\lambda \in B(A)$) is called a F-left (right, bi-) singular word in A. If $L(A) \neq 0$, ($R(A) \neq 0$, $B(A) \neq 0$), then A is called a fuzzy left (right, bi-)singular language.

**Theorem 2.1.** If A is a fuzzy left (right, bi-) singular language $\Leftrightarrow A_{\dot{o}}$ is an ordinary left (right, bi-) singular language.

**Proof.** If A is a fuzzy left singular language, then $L(A)\neq 0$, i.e. there exists $g\in X^+$ such that $(A\text{-}A X^+)(g)>0$ and $A(gx)=0$ for all $x\in X^+$. So $A(g)>0$, $(A X^+)(g)=0$ and $A(gx)=0$ for all $x\in X^+$. Thus $g\in A_{\dot{o}}$, $gx\notin A_{\dot{o}}$, for all $x\in X^+$ and $g = yz$, $z\in X^+$ implies $y\notin A_{\dot{o}}$. Hence $g\in l(A_{\dot{o}})$ and $A_{\dot{o}}$ is an ordinary left singular language.

Conversely, if $A_{\dot{o}}$ is an ordinary left singular language, then $l(A_{\dot{o}})\neq \phi$. So there exists $g\in A_{\dot{o}}$ such that $gx \notin A_{\dot{o}}$ for all $x\in X^+$ and $g = yz$, $z\in X^+$ implies $y\notin A_{\dot{o}}$, i.e. $A(g)>o$, $(A X^+)(g)\leqslant 0$ and $A(gx)\leqslant 0$ for all $x\in X^+$. So $L(A)(g)=(A\text{-}AX^+)(g)=A(g)>0$, i.e. $L(A)\neq 0$. Hence A is a fuzzy left singular language.

Similarly, we can obtain other proof.

**Theorem2.2.** If $0\neq A\in F(X^+)$, set $M=Sup\{L(A)(g)\mid g\in X^+\}$ and $L(A)\neq 0$. then A is a fuzzy left (right, bi-) singular language iff $A_{\dot{\lambda}}$ is an ordinary left (right, bi-) singular language, $0\leqslant \lambda <M$

**Proof.** By theorem2.1 it is neccessary to prove that $0\leqslant\lambda<M$, $[L(A)]_{\lambda}\subseteq l(A_{\lambda})$($[R(A)]_{\lambda}\subseteq r(A_{\lambda})$, $[B(A)]_{\lambda}\subseteq b(A_{\lambda})$). In fact, $0\leqslant\lambda<M$, $g\in[L(A)]_{\lambda}$, then $L(A)(g)>\lambda$, i.e. $A(g)>\lambda$, $(AX^+)(g)=0$ and $A(gx)=0$ for all $x\in X^+$, therefore $g\in A_{\lambda}$ $gx\notin A_{\lambda}$, for all $x\in X^+$ and $g=yz$, $z\in X^+$ implies $y\notin A_{\lambda}$, i.e. $g\in l(A_{\lambda})$ and $[L(A)]_{\lambda}\subseteq l(A_{\lambda})$.

**Corollary2.3.** If A is a fuzzy left singular language $\Leftrightarrow$ $suppL(A)\neq 0$.

**Proof.** It is easy to prove $[L(A)]_{\dot{0}}=l(A_{\dot{0}})$. By proposition 2.2, we can obtain the proof.

**Theorem 2.4.** Let $0\neq A\in F(X^+)$, $\lambda',\lambda \in(0,1]$, $g_{\lambda}\in A$, then A fuzzy point $g_{\lambda}\in L(A)$ ($g_{\lambda}\in R(A)$, $g_{\lambda}\in B(A)$) if and only if for any fuzzy point $x_{\lambda'}\in A$, $\{g_{\lambda},x_{\lambda'}\}$ (i.e. $g_{\lambda}\vee x_{\lambda'}$) is a F-prefix (suffix, bifix)code.

**Proof.** Suppose $g_{\lambda}\in L(A)$ and there exists $x_{\lambda'}\in A$ such that $\{g_{\lambda}, x_{\lambda'}\}$ is not a fuzzy prefix code. Then there exists $u\in X^+$ such that $\{g_{\lambda}, x_{\lambda'}\}(u)>0$ and $(\{g_{\lambda}, x_{\lambda'}\}X^+)(u)=\underset{\substack{y,z\in X^+\\ yz=u}}{SUP}\{g_{\lambda}, x_{\lambda'}\}(y)>0$, i.e. $\{g_{\lambda}, x_{\lambda'}\}(u)>0$ and there exists $y, z\in X^+$ such that $yz=u$, $\{g_{\lambda}, x_{\lambda'}\}(y)>0$. For $\{g_{\lambda}, x_{\lambda'}\}(u)>0$, we have $u=g$ or $u=x$. If $u=g$, since $\{g_{\lambda}, x_{\lambda'}\}(y)>0$ and $yz=u$, $z\in X^+$, then $y=x$ and $xz=g$. So $(A X^+)(g)\geqslant A(x)\geqslant\lambda'>0$ and hence $(A\text{-}A X^+)(g)=0$. Which contradicts to that $g_{\lambda}\in L(A)$. If $u=x$, since $\{g_{\lambda}, x_{\lambda'}\}(y)>0$ and $yz=u$, $z\in X^+$, then $y=g$ and $gz=x$. So $A(gz)=A(x)\geqslant\lambda'>0$ and hence $L(A)(g)=0$. Which contradicts to that $g_{\lambda}\in L(A)$. Therefore $\{g_{\lambda}, x_{\lambda'}\}$ is a fuzzy prefix code for any $x_{\lambda'}\in A$.

Conversely, suppose $\{g_{\lambda}, x_{\lambda'}\}$ is a F-prefix code for any $x_{\lambda'}\in A$.If there exists $y\in X^+$ such that $A(gy)>0$, then let $x=gy$, $\lambda'=A(gy)>0$ and $\{g_{\lambda}, x_{\lambda'}\}$ is not a F-prefix code, a contradiction. If $(AX^+)(g)>0$, then there exists $y, z\in X^+$ such that $g=yz$, $A(y)>0$. So let $x=y$, $\lambda'=A(y)>0$ and hence $\{g_{\lambda}, x_{\lambda'}\}$ is not F-prefix code, a contradiction. Therefore $AX^+(g)=0$ and $A(gy)=0$ for all $y\in X^+$. Hence $L(A)(g)=(A\text{-}A X^+)(g)=A(g)\geqslant\lambda>0$ and $g_{\lambda}\in L(A)$.

Similarly, we can obtain $\{g_{\lambda}, x_{\lambda'}\}$ is a fuzzy suffix (bifix) code iff $g_{\lambda}\in R(A)$ ($g_{\lambda}\in B(A)$).

**Theorem2.5.** If $0\neq A\in F(X^+)$, then A is a F-prefix (suffix, bifix) code if and only if

L(A)=A (R(A)=A,B(A)=A).

**Proof.** Suppose A is a F-prefix code. Then $A\cap AX^{+}=\phi$. For any $g\in X^{+}$ if A(g)=0, then $L(A)(g)\leqslant(A-A\ X^{+})(g)\leqslant A(g)=0$; if A(g)>0, since $A\cap AX^{+}=\phi$, then $(AX^{+})(g)=0$. Since $(AX^{+})(gx)=\sup\{A(y)\mid y,z\in X^{+}, yz=gx\}\geqslant A(g)>0$ and $A\cap AX^{+}=\phi$, then A(gx)=0, for all $x\in X^{+}$. Thus $L(A)(g)=(A-A\ X^{+})(g)=A(g)$. Therefore for any $g\in X^{+}$, we have L(A)(g)=A(g), i.e. L(A)=A.

Conversely, suppose A=L(A) and A is not a F-prefix code. Then there exists $g\in X^{+}$ such that A(g)>0 and $(AX^{+})(g)>0$. So $0<A(g)=L(A)(g)\leqslant(A-AX^{+})(g)=0$, a contradiction.

Similarly, we can dtain that A is a F-suffix (bifix) code if R(A)=A (R(A)=A).

Let $S_l$ (Sr, $S_b$) be the class of all fuzzy left. (rihght, bi-) singular languages in $F(X^{*})$, and let $\{(1,1)\}\in S_l$ (Sr, $S_b$).

**Proposition2.6.** If fuzzy point $g_{\lambda}\in L(A)$ and $h_{\lambda'}\in L(B)$, then $(gh)_{\bar{\lambda}}\in L(AB)$, where $\bar{\lambda}=\min(\lambda,\lambda')$.

**Proof.** Let $u_{\lambda_1}\in A, v_{\lambda_1'}\in B$ and $(uv)_{\bar{\lambda}_1}\in AB$, where $\overline{\lambda_1}=\min(\lambda_1,\lambda_1')$. Then by Theorem 2.4, $\{g_{\lambda},u_{\lambda_1}\}$ and $\{h_{\lambda'},v_{\lambda_1'}\}$ are fuzzy prefix codes. Since fuzzy prefix codes are closed under concatenation (see [4]), $\{g_{\lambda},u_{\lambda_1}\}.\{h_{\lambda'},u_{\lambda_1'}\}=\{(gh)_{\bar{\lambda}},(gv)_{\bar{\lambda}_2},(uh)_{\bar{\lambda}_3},(uv)_{\bar{\lambda}_1}\}$ is a fuzzy prefix code, where $\bar{\lambda}_2=\min(\lambda,\lambda_1')$, $\bar{\lambda}_3=\min(\lambda_1,\lambda')$. We then have $\{gh)_{\bar{\lambda}},(uv)_{\bar{\lambda}_1}\}$ is a F-prefix code. This shows that $(gh)_{\bar{\lambda}}\in L(AB)$.

**Remark1.** Similar to the proof of proposition 2.6, we can show that $g_{\lambda}\in R(A)$ and $h_{\lambda'}\in R$ (B) imply $(gh)_{\bar{\lambda}}\in R(AB)$ and that $g_{\lambda}\in B(A)$ and $h_{\lambda'}\in B(B)$ imply $(gh)_{\bar{\lambda}}\in B(AB)$, where $\bar{\lambda}=\min(\lambda,\lambda')$.

From the above results, we have

**Proposition2.7.** $S_l$, Sr, and $S_b$ are submonoids of $F(X^{*})$.

**Proof.** By Proposition2.6, we have that the elements of $S_l$ are closed under concatenation, also as fuzzy languages are associative under concatenation. Hence $S_l$ is a submonoid of $F(X^{*})$.

Similar we can obtain Sr and $S_b$ are submonoids of $F(X^{*})$

## 3. FUZZY STRONGLY SINGULAR LANGUAGE

**Definition3.1.** A fuzzy language $A\in S_l$ (Sr, $S_b$) is called a fuzzy strongly left (right, bi-) singular language if $L(A)\supseteq\underline{A}$ ($R(A)\supseteq\underline{A}$, $B(A)\supseteq\underline{A}$).Where $\underline{A}$ is defined by.

$$\underline{A}(x)=\begin{cases}A(x) & x\in x^{+} \text{ and } \lg(x)\le\lg(y) \text{ for all } y\in SuppA\\ 0 & otherwise\end{cases}$$

Let $S_l'(S_r',S_b')$ to be the corresponding class of all fuzzy strongly left (right, bi-) singular languages on $F(X^{*})$.

**Theorem.3.1.** The class of all fuzzy strongly left (right, bi-) singular languages

$S_l'\left(S_r', S_b'\right)$ is a submonoid of F(X*)

**Proof.** Since $\underline{A} \subseteq L(A), \underline{B} \subseteq L(B)$ and $\underline{AB} = \underline{AB} \subseteq L(A)L(B) \subseteq L(AB)$, and since fuzzy languages are associative under concatenation. So $S_l'$ is a submonoid of F(X*).

Similarly, we obtain $S_r'$ and $S_b'$ are submonoids of F(X*).

**Proposition 3.2.** The monoid $S_l'$ is a sp submonoid of F(X*).

**Proof.** We want to show that AB∈ $S_l'$ implies that B∈ $S_l'$ for all A,B∈F(X*). We may assume A≠{(1,1)}, B≠{(1,1)}.It suffices to show that for $\underline{b}_\lambda \in \underline{B}, \underline{b}_\lambda$ is a F-left singular word in B. Let x∈X* such that $B(\underline{b}x) > 0$, then (AB)($\underline{ab}x$) > 0 for all $\underline{a} \in x^+$ such that A($\underline{a}$)>0. Since $\underline{AB} = \underline{AB} \subseteq L(AB)$. We have 0<($\underline{AB}$)($\underline{ab}$) ≤ $L(AB)(\underline{ab})$), it follows that x=1, because AB∈ $S_l'$ .

**Proposition3.3.** $S_r$ is a strongly suffix submonoid of F(X*)=M.

**Proof.** Let B∈ $S_r M^{-1}$.Then there exists A∈M such that BA∈ $S_r$. So there exists $b_\lambda$ ∈B, $a_{\lambda'}$ ∈A such that (BA)(ba)>0. If there exists $b' \in$ suppB and $x \in X^+$ such that $b = xb'$ or $b' = xb$, then $ba = xb'a$ or $b'a = xba$ .Thus (BA)(ab)=0, a contradiction. So $b_\lambda \in R(B)$ and $R(B) \neq \phi$ ,i.e. $B \in S_r$ .Therefore $S_r M^{-1} \subseteq S_r$.

**Proposition 3.4.** $S_b'$ is a strongly suffix submonoid of $F_l = S_l' \cap S_r'$.

**Proof.** Let $A, B \in F_l$ such that $AB \in S_b'$ , i.e. $\underline{AB} \subseteq L(AB) \cap R(AB)$. We want to show that $A \in S_b'$ , i.e. We want to see that $\underline{A} \subseteq R(A)$, because $F_l \subseteq S_l'$ implies $\underline{A} \subseteq L(A)$.

Indeed, if $a_0$ ∈supp $\underline{A}$ , x∈X+ and a ∈SuppA such that $a_0 = xa$ or $a = xa_0$, therefore $b_0 \in$ supp $\underline{B}$, $a_0 b_0 = xab_0$ or $ab_0 = xa_0 b_0$, and hence $R(AB)(a_0 b_0) = 0$. So $0 < (\underline{AB})(a_0 b_0) = (\underline{AB})(a_0 b_0) \leq R(AB)(a_0 b_0) = 0$, a contradiction. Therefore $a_0 \neq xa$ and $a \neq xa_0$, hence $R(A)(a_0) = A(a_0) > 0$ and $\underline{A} \subseteq R(A)$. Thus $A \in S_b'$ .

**Remark2.** Similar to the proof of the Proposition, we can obtain that $S_b'$ is a strongly prefix submonoid of $F_r = S_r' \cap S_l$.

## 4. Cancellative property

**Definition4.1.** Let S be a semigroup and

$$D_\lambda(S) = \left\{ A \in F(S) \middle| AB = AC \ \ implies \ \ B_\lambda = C_\lambda \ \ for \ all \ \ B, C \in F(S) \right\}, \text{where } \lambda \in [0,1].$$

If $D_\lambda$(S)≠ Φ, then any element of $D_\lambda$(S) is called a λ-left cancellative element of F(S). In particular, an o-left cancellative element of F(S) is called a Fuzzy left cancellative element of F(S).

**Theorem4.1.** Let A ∈F(X*), A≠{(1,1)}, if A contains a F-left singular word $g_{\lambda'}$,then for any λ ∈ [0,A(g)) , A is a left $\lambda$-cancellative element in F(X*).

**Proof.** Let AB=AC, where B,C∈F(X+).. (We note that B={(1,1)} if and only if C={(1,1)}). We want to show that B $_\lambda$ =C $_\lambda$ .Let x ∈ X+such that B(x) > 0 min

(A(g),B(x))=sup min $_{g',x'\in X', gx=g'x'}(A(g')B(x') = (AB)(gx) = AC(gx) = \sup\min_{y,z\in X^*, g=yz}(A(y),C(z))$.
=min(A(g),C(x)). So if *B(x)<A(g)* then B(x)=c(x). If *B(x)≥A(g)* then *C(x)≥A(g),* therefore $\underset{\bullet}{B_\lambda} = \underset{\bullet}{C_\lambda}$ for any $\lambda \in [0,A(g)]$.

**Corollary4.2.** Let $A \in F(X^+)$ be a fuzzy code, then $A \in D_\lambda(X^+)$, where $0 \le \lambda < Sup_{x\in X^+} A(x)$.

**Theorem4.3.** Let $A,C \in S_l', B,D \in F(X^*)$, AB=CD

(1) If $l_g$(suppA)=$l_g$(suppC), then $\underset{\bullet}{B_0} = \underset{\bullet}{D_0}$

(2) If there exists $g \in X^+$ such that $A(g) = C(g) = \lambda' > 0$ and $l_g$g=$l_g$(suppA)= $l_g$(suppC), then $\underset{\bullet}{B_\lambda} = \underset{\bullet}{D_\lambda}$, $\lambda \in [0,\lambda')$

**Proof.** (1)Since AB=CD and $l_g$(suppA)=$l_g$(suppC), by proposition of [8], we have supp$\underline{A}$=supp$\underline{C}$ and supp$\underline{B}$=supp$\underline{D}$. Now let a$\in$supp$\underline{A}$=supp$\underline{C}$. Then for any $b \in \underset{\bullet}{B_0}$, (CD)(ab)=(AB)(ab)>0. Since $C \in S_l'$, then $\underline{C} \subseteq L(C)$ and hence $(CD)(ab) = \sup\{\min(C(a'),D(b'))| a',b' \in X^*, a'b' = ab\} = \min(C(a),D(b)) > 0$ .So $D(b) > 0$, i.e. $b \in \underset{\bullet}{D_0}$ and $\underset{\bullet}{B_0} \subseteq \underset{\bullet}{D_0}$.Similarly, we have $\underset{\bullet}{B_0} \supseteq \underset{\bullet}{D_0}$ . Hence $\underset{\bullet}{B_0} = \underset{\bullet}{D_0}$

(2) It is obvious that $g \in$supp$\underline{A}$=supp$\underline{C}$. For any $b \in X^*$, since $A,C \in S_l'$, then $\underline{A} \subseteq L(A)$, $\underline{C} \subseteq L(C)$ and $\min(C(g),D(b)) = (CD)(gb)$ =(AB)(gb)= $\min(A(g)\ B(b))$.

So $B(b) \ge A(g) \Leftrightarrow D(b) \ge A(g)$ or $B(b) \le A(g) \Rightarrow D(b) = B(b)$. Therefore for any $\lambda \in [0,\lambda'), \underset{\bullet}{B_\lambda} = \underset{\bullet}{D_\lambda}$.

**Corollary4.4.** (1) Let A,B,C,D$\in F(X^*)$. If AB=CD and $g_{\lambda'}$ is a common F-left singular word of A and C, then $\underset{\bullet}{B_\lambda} = \underset{\bullet}{D_\lambda}$ for any $\lambda \in [0,\lambda')$.

(2) Let $A,C \in S_l', B,D \in F(X^*)$. If AB=CD and $\underline{A} = \underline{C}$, then $\underset{\bullet}{B_\lambda} = \underset{\bullet}{D_\lambda}$ for any $\lambda \in [0, \underset{x\in X^*}{Sup} A(x))$.

**Proposition4.5.** Let AB=AC,A,B,C ∈ F(x*),then $A(B \cup D) = A(C \cup D)$ for all $D \in F(X^*)$.

The proof is omitted.

**Theorem4.6.** Let $A \in F(X^*)$, then $A \notin \underset{\bullet}{D_\lambda}(X^*)$ if and only if $\underset{\bullet}{A_\lambda} X^+ = \underset{\bullet}{A_\lambda} X_X^+$ for some $x \in X^+$, where $X_X^+ = X^+ \setminus \{x\}$.

**Proof.** ($\Leftarrow$) For any $y \in X^+$ such that $yx \in A_{\dot{\lambda}} X^+$, if $A(y) > \lambda$, then $yx \in A_{\dot{\lambda}} X^+ = A_{\dot{\lambda}} X_X^+$, which contradicts to that $yx \notin A_{\dot{\lambda}} X_X^+$. So $y \notin A_{\dot{\lambda}}$ and $A(y) \le \lambda$. Let $B(u) = C(u)$ if $u \ne x$ or $yu \notin A_{\dot{\lambda}} X^+$ for all $y \in X^+, B(u) > \lambda$ and $C(u) = \lambda$ if $u = x$ and $yu \in A_{\dot{\lambda}} X^+$ for all $y \in X^+$. For any $h \in X^+$, if $h \ne yx$ for all $y \in X^*$, then

$$(AB)(h) = \underset{\substack{y,z \in X^* \\ yz = h}}{Sup} \min(A(y), B(z)) = \underset{\substack{y,z \in X^* \\ yz = h}}{Sup} \min(A(y), C(z)) = (AC)(h).$$

If $h = yx$ for some $y \in X^*$, then $A(y) \le \lambda$ for $yx \in A_{\dot{\lambda}} X^+$

$$(AB)(h) = \underset{\substack{y,z \in X^* \\ yz = h}}{Sup} \min(A(y), B(z)) = \max\{ \underset{\substack{y,z \in X^* \\ z \ne x \\ yz = h}}{Sup} \min(A(y), B(z)), \min(A(y), B(x))\}$$

$$= \max\{ \underset{\substack{y,z \in X^* \\ z \ne x \\ yz = h}}{Sup} \min(A(y), C(z)), \min(A(y), C(z))\} = \underset{\substack{y,z \in X^* \\ yz = h}}{Sup} \min(A(y), B(z))$$

$=(AC)(h)$. So AB=AC and $B_{\dot{\lambda}} \ne C_{\dot{\lambda}}$ (since $x \in B_{\dot{\lambda}}, x \notin C_{\dot{\lambda}}$). Therefore $A \notin D_\lambda(X^*)$.

($\Rightarrow$) Suppose $A \notin D_\lambda(X^*)$, then there exists $B, C \in F(X^*)$ such that $B_{\dot{\lambda}} \ne C_{\dot{\lambda}}$ and $AB=AC$, we may assume $x \in B_{\dot{\lambda}}, x \notin C_{\dot{\lambda}}$ for some $x \in X^+$. Then by Proposition3.6, $A(B \cup X_X^+) = A(C \cup X_X^+)$. Since $x \in B_{\dot{\lambda}}$ and $x \notin C_{\dot{\lambda}}$, then $B_{\dot{\lambda}} \cup X_X^+ = X^+$ and $C_{\dot{\lambda}} \cup X_X^+ = X_X^+$. So $[A(B \cup X_X^+)]_{\dot{\lambda}} = [A(C \cup X_X^+)]_{\dot{\lambda}}$, i.e. $A_{\dot{\lambda}} X^+ = A_{\dot{\lambda}} X_X^+$.

**Proposition4.7.** Let $u \in X^+$, if $P_A X^+ \subseteq AX^+$, then $A_{\dot{0}} X^+ = A_{\dot{0}} X_u^+$, where $P_A = A - AX^+$

**Proof.** Let $h \in X^+$ such that $(A - P_A)(h) > 0$, then $h = py$ for some $P_A(p) > 0$ and $y \in X^+$. It follows that $hu = (py)u = p(yu)$. Since $P_A X^+ \subseteq AX_u^+$, and $(AX_u^+)(hu) = \sup\{A(x) | x \in X^*, z \in X_u^+, xz = hu\} \ge A(p) > 0$, then $A_{\dot{0}} X^+ \subseteq A_{\dot{0}} X_u^+$. Therefore $A_{\dot{0}} X^+ = A_{\dot{0}} X_u^+$.

**Proposition4.8.** Let $A \in F(X^+)$, if $P_A X^+ \subseteq AX_u^+$, then $A \notin D_0(X^*)$.

The proof is omitted.

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