TRIANGULAR NORMS AND MV-ALGEBRAS

JÁN RYBÁRIK

ABSTRACT. MV-algebras on real chains are discussed. A finite chain admits unique MV-algebra. The interval chain admits the MV-algebras isomorphic with the bold fuzzy MV-algebra only.

1. INTRODUCTION.

MV-algebras were introduced in 1958 by Chang [2], see also [1,3]. In the same year triangular norms were defined by Schweizer and Sklar [16], see also [6,7,9,17]. The aim of this paper is the investigation of relationship between t-norms and MV-algebras which are defined pointwisely on some universe $\mathbf{\Omega}$.

Definition 1.1 [2,3]. An algebra $\{ \mathcal{M}, \mathbf{0}, \mathbf{1}, ', \oplus, \odot \}$ is said to be **an MV-algebra** iff it satisfies the following equations:

(MV1)	$(x \oplus y) \oplus z = x \oplus (y \oplus z)$	(*associativity*)
(MV2)	$x \oplus y = y \oplus x$	(*symmetry*)
(MV3)	$x \oplus \mathbf{0} = x$	(*neutral element*)
(MV4)	$x \oplus \mathbf{1} = \mathbf{1}$	(*annihilator*)
(MV5)	$\mathbf{0}' = \mathbf{1}$ *and* $\mathbf{1}' = \mathbf{0}$	(*boundary conditions*)
(MV6)	$x \odot y = (x' \oplus y')'$	(*De Morgan law*)
(MV7)	$y \oplus (y \oplus x')' = x \oplus (x \oplus y')'$	(*compatibility*).

Note that (MV7) ensures $(x')' = x$ for all $x \in \mathcal{M}$, i.e., the complementation $' : \mathcal{M} \to \mathcal{M}$ is an involutive mapping. In addition, by De Morgan law (MV6), the operation $\odot$ is associative and symmetric, with the neutral element $\mathbf{1}$ and annihilator $\mathbf{0}$. Further, the compatibility allows to introduce the lattice structure on $\mathcal{M}$: $x \vee y = x \oplus (x \oplus y')'$
and the partial order $\leq$: $x \leq y$ iff $x \vee y = y$.
For more details and other properties of MV-algebras we recommend the overview paper of Cignoli and Mundici [3].

It is evident that if $\mathcal{M}$ is a chain (with respect to $\leq$), i.e., $\leq$ is a total order, then both operations $\oplus$ and $\odot$ are non-decreasing.

1991 *Mathematics Subject Classification.* 0521.
Key words and phrases. triangular norm, nilpotent t-norm, MV-algebra, semisimple MV-algebra.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Recall that an MV-algebra $\mathcal{M}$ is called **semisimple** (or Archimedean), see [1,10], iff for all $x \neq \mathbf{0}$ there is $n \in \boldsymbol{N}$ such that $nx = \underbrace{x \oplus \cdots \oplus x}_{n-times} = \mathbf{1}$.

By the duality, for all $x \neq \mathbf{1}$ there is $n \in \boldsymbol{N}$ such that $x^n = \underbrace{x \odot \cdots \odot x}_{n-times} = \mathbf{0}$.

Definition 1.2. Let $(\boldsymbol{L}, \leq, \mathbf{0}_L, \mathbf{1}_L)$ be a lattice with a partial order $\leq$, the minimal element $\mathbf{0}_L$ and the maximal element $\mathbf{1}_L$. A binary operation $T : \boldsymbol{L}^2 \to \boldsymbol{L}$ is called **a triangular norm** iff it is

(1) symmetric, $T(u,v) = T(v,u)$

(2) non-decreasing, $u \leq u',\ v \leq v' \Longrightarrow T(u,v) \leq T(u',v')$

(3) associative, $T(u,T(v,w)) = T(T(u,v),w)$

(4) $\mathbf{1}_L$ is the neutral element, $T(u,\mathbf{1}_L) = u$.

Note that the above definition from [4], see also [5,12], generalizes the standard definition [16,17] of Schweizer and Sklar, which is, indeed, a special case when $\boldsymbol{L} = [\,0,\ 1\,]$.

In what follows, we will restrict ourselves to the case when $\boldsymbol{L}$ is a finite real set or a closed extended real interval and $\leq$ is the usual order of reals.

The relationship between MV-algebras and triangular norms was observed first (indirectly) by Belluce [1], who showed that any semisimple MV-algebra $\mathcal{M}$ is isomorphic with some bold MV-algebra, i.e., an MV-algebra of fuzzy subsets of some universe $\boldsymbol{\Omega}$ (corresponding to the maximal ideals of $\mathcal{M}$) equipped with $\odot$ corresponding to the Lukasiewicz t-norm T_L, $T_L(u,v) = \max(0,\ u+v-1)$, $u' = 1-u$ and $\oplus$ corresponds to the dual operator S_L (Lukasiewicz t-conorm), $S_L(u,v) = \min(1,\ u+v)$. Another investigation on the relationship of t-norms and MV-algebras were done in the framework of difference posets [8] by Mesiar and Pap [10,11,13].

In the present paper we will discuss another point of view. Namely, let $\boldsymbol{L}$ be either a finite chain, $\boldsymbol{L} = \{\,x_0,\ x_1, \ldots, x_n\,\}$, $n \in \boldsymbol{N}$, $0 = x_0 < x_1 < \ldots < x_n = 1$ or $\boldsymbol{L} = [\,0,\ 1\,]$. The general case when $\boldsymbol{L}$ is a finite subset of $\bar{\boldsymbol{R}}$ or the interval $[\,a,\ b\,] \subset \bar{\boldsymbol{R}}$ is a matter of rescaling only.

For a given universe $\boldsymbol{\Omega} \neq \emptyset$, we will deal with the system $\mathcal{M} \equiv \boldsymbol{L}^{\boldsymbol{\Omega}}$ of all $\boldsymbol{L}$-valued mappings defined on $\boldsymbol{\Omega}$, equipped with pointwisely defined order $\leq$ (partial order if card $\boldsymbol{\Omega} > 1$), a unary operation $'$ and binary operations $\oplus$, $\odot$, i.e., for $f,g \in \mathcal{M}$ and $\omega \in \boldsymbol{\Omega}$ we have

$$' : \boldsymbol{L} \to \boldsymbol{L}, \qquad f'(\omega) = (f(\omega))'$$

$$\oplus : \boldsymbol{L}^2 \to \boldsymbol{L}, \qquad (f \oplus g)(\omega) = f(\omega) \oplus g(\omega)$$

$$\odot : \boldsymbol{L}^2 \to \boldsymbol{L}, \qquad (f \odot g)(\omega) = f(\omega) \odot g(\omega).$$

The use the same notation for the above mentioned operations on $\boldsymbol{L}$ and on $\mathcal{M}$ cannot cause any discrepancy.

We are interested under which the circumstances $(\mathcal{M},\ \leq,\ ',\ \oplus,\ \odot)$ is an MV-algebra. It is evident that due to its pointwise structure, $(\mathcal{M},\ \leq,\ ',\ \oplus,\ \odot)$ is an MV-algebra if and only if $(\boldsymbol{L},\ \leq,\ ',\ \oplus,\ \odot)$ is an MV-algebra (with the usual order on the real line).

2. Finite case.

Let $L = \mathcal{C} = \{ x_0, x_1, \ldots, x_n \}$ be a finite (ordered) set of reals, $0 = x_0 < x_1 < \ldots < x_n = 1$. Suppose that there is an MV-algebra $(\mathcal{C}, 0, 1, ', \oplus, \odot)$ such that the corresponding order $\leq$ is the usual order of reals (from $\mathcal{C}$).

Lemma 2.1. *There is the unique involutive complementation* $' : \mathcal{C} \to \mathcal{C}$.

Proof. By [3], the mapping $'$ is non-increasing, i.e. $x \leq y \Rightarrow y' \leq x'$. Due to involutivity of complementation and finiteness of $\mathcal{C}$, the only appropriate complementation is given by $x_i' = x_{n-i}$. □

Lemma 2.2. *The operation* $\odot : \mathcal{C}^2 \to \mathcal{C}$ *is a t-norm.*

Proof. As were mentioned in Section 1, the operation $\odot$ is symmetric, non-decreasing, associative and with neutral element 1. Thus by Definition 1.2, $\odot$ is a t-norm. □

Lemma 2.3. *The operation* $\odot$ *is determined uniquely by* $x_i \odot x_j = x_{i+j-n}$.

Proof. By the duality, in any MV-algebra $\mathcal{M}$ the equality
$x \vee y = x \oplus (x \oplus y')'$ is equivalent with the equality $x \wedge y = x \odot (x \odot y')'$.
But then $0 = x \odot x'$ for all $x \in \mathcal{M}$.
For $\mathcal{M} = \mathcal{C}$ this means that $x_i \odot x_{n-i} = 0$ and due to monotonicity of $\odot$ we obtain $x_i \odot x_j = 0 = x_0$ whenever $i + j \leq n$.
For any $i \in \{0, 1, \ldots, n-1\}$ we have $x_i = x_{n-1} \wedge x_i = x_{n-1} \odot (x_{n-1} \odot x_{n-i})'$.
Then the sequence $\{x_{n-1} \odot x_{n-i}\}_{i=0}^{n-1}$ should be strictly decreasing. More, $x_{n-1} \odot 1 = x_{n-1} \odot x_{n-0} = x_{n-1}$ and $x_0 = 0 = x_{n-1} \odot x_1 = x_{n-1} \odot x_{n-(n-1)}$.
Therefore $x_{n-1} \odot x_{n-i} = x_{n-i-1} = x_{n-1+n-i-n}$ whenever $n-1+n-i-n \geq 0$.
The rest of the proof can be shown in a similar manner. □

Summarizing all above results, we see that there is unique MV-algebra on $\mathcal{C}$.

Theorem 2.1. *Let* $\mathcal{C} = \{ x_0, x_1, \ldots, x_n \}$ *be a finite real chain. Then there is unique MV-algebra on* $\mathcal{C}$ *with operations* $\oplus, \odot, '$ *given by*

$$\begin{aligned} x_i \oplus x_j &= x_{\min(n,i+j)}, \\ x_i \odot x_j &= x_{\max(0,i+j-n)}, \\ x_i' &= x_{n-i}, \end{aligned}$$

respectively.

Remark 2.1. An alternative proof follows from the Belluce's result [1]. By (MV7), the only idempotents of $\oplus$ can be the elements $x_0 = 0$ and $x_n = 1$. Consequently, any MV-algebra on a finite chain $\mathcal{C}$ is Archimedean. The only maximal ideal on $\mathcal{C}$ is $\mathcal{C}$ itself and hence $\mathcal{C}$ is isomorphic with a system of fuzzy subsets of some singleton equipped with bold connectives, i.e., there is a mapping $g : \mathcal{C} \to [0, 1]$ such that

$$\begin{aligned} g(x_i \oplus x_j) &= \min(1, g(x_i) + g(x_j)) = g(x_i) \boxplus g(x_j), \\ g(x_i \odot x_j) &= \max(0, g(x_i) + g(x_j) - 1) = g(x_i) \boxdot g(x_j), \\ g(x_i') &= 1 - g(x_i) = n(g(x_i)). \end{aligned}$$

It is evident that unique mapping g fulfilling these requirements will be given by $g(x_i) = \frac{i}{n}$. Then $x_i \odot x_j = g^{-1}(\max(0, \frac{i}{n} + \frac{j}{n} - 1)) = x_{\max(0,i+j-n)}$.

This same result follows also from uniqueness of a difference poset on a finite chain shown by Riečanová and Bršel [15].

3. The continuous case $L = [0, 1]$.

Let $([0, 1], 0, 1, ', \oplus, \odot)$ be an MV-algebra (with the usual order of reals). By Trillas[18], see also [7,14], we have the following characterization of involutive complementations on $[0, 1]$.

Lemma 3.1. *The unary operation* $' : [0, 1] \to [0, 1]$ *is an MV-algebra's complementation iff there is an increasing bijection* $g : [0, 1] \to [0, 1]$ *such that for all* $x \in [0, 1]$ *holds*

$$x' = g^{-1}(1 - g(x)).$$

Note that the function g is called a generator of this complementation and that two generators g and h generate the same complementation if and only if their composition $h \circ g^{-1}$ generates the standard complementation n, $n(x) = 1 - x$ for all $x \in [0, 1]$.

Similarly as in the finite case we obtain the next result.

Lemma 3.2. *The operation* $\odot : [0, 1]^2 \to [0, 1]$ *is a t-norm.*

Lemma 3.3. *The operation* $\odot : [0, 1]^2 \to [0, 1]$ *is a nilpotent t-norm, i.e., there is a decreasing bijection* $f : [0, 1] \to [0, 1]$ *such that*

$$x \odot y = f^{-1}(\min(1, f(x) + f(y))).$$

Proof. The monotonicity of the operation $\odot$ and (MV7) ensure the continuity of the t-norm $\odot$. Indeed, for any $y \le x$ there is z $(= (x \odot y')')$ such that $x \odot z = y$, what together with monotonicity implies the continuity of the partial mapping $(x \odot .)$. By [6,7], the joint continuity of $\odot$ follows.

Now, suppose that $x \odot x = x$, i.e., x is an idempotent element of $\odot$. Then $x \wedge x' = x \odot (x \odot x'')' = x \odot x' = 0$ and thus $x \in \{0, 1\}$. By [6,9,17], $\odot$ is a continuous Archimedean t-norm. However, $x \odot x' = 0$ for all $x \in [0, 1]$ excludes the strict t-norms, and hence $\odot$ is a nilpotent t-norm. The existence of its additive normed generator f (which is unique) is a standard result from [6,9,17]. □

Finally, we are able to prove the main result of this section.

Theorem 3.1. *The system* $([0, 1], 0, 1, ', \oplus, \odot)$ *is an MV-algebra if and only if it is isomorphic with the bold MV-algebra* $([0, 1], 0, 1, n, \boxplus, \boxdot)$, *i.e., there is an increasing bijection* $h : [0, 1] \to [0, 1]$ *such that*

$$x' = h^{-1}(1 - h(x)),$$
$$x \oplus y = h^{-1}(h(x) \boxplus h(y)) = h^{-1}(\min(1, h(x) + h(y))),$$
$$x \odot y = h^{-1}(h(x) \boxdot h(y)) = h^{-1}(\max(0, h(x) + h(y) - 1)).$$

Proof. The if part of this theorem is obvious.
What concerns the only part of the theorem, we apply Lemma 3.3. Then the operation $\oplus$ is a nilpotent t-conorm [6,9,14,17] and hence it is generated by unique normed generator $h : [0, 1] \to [0, 1]$, h is an increasing bijection, $x \oplus y = h^{-1}(\min(1, h(x) + h(y))$.
Then for all $x \in [0, 1]$ we get
$1 = x \oplus x' = h^{-1}(\min(1, h(x) + h(x'))$, i.e., $h(x) + h(x') \geq 1$.
The continuity of h excludes the strict inequality $h(x) + h(x') > 1$ (for any $x \in]0, 1[$), otherwise we will get $x \oplus u = 1$ for some $u < x'$ obtaining a contradiction. However, then for all $x \in [0, 1]$ it is
$1 = h(x) + h(x') = h(x) + h(g^{-1}(1 - g(x))) = h \circ g^{-1}(u) + h \circ g^{-1}(1 - u)$,
where g is some generator of the complementation $'$ and $u = g(x)$. But this means that h is also a generator of $'$.
Finally,

$$\begin{aligned} x \odot y &= (x' \oplus y')' = h^{-1}(1 - h(x' \oplus y')) = \\ &= h^{-1}(1 - h \circ h^{-1}(\min(1, h(x') + h(y')))) = \\ &= h^{-1}(1 - \min(1, h \circ h^{-1}(1 - h(x)) + h \circ h^{-1}(1 - h(y)))) = \\ &= h^{-1}(1 - \min(1, 2 - h(x) - h(y))) = h^{-1}(\max(0, h(x) + h(y) - 1)). \end{aligned}$$

□

4. Conclusions.

We have shown the simple structure of MV-algebras on finite chains (the case isomorphic to the finite real chains) and on the real intervals $[a, b]$. In both cases, the corresponding MV-algebra is Archimedean (semisimple) and hence we can apply the Belluce representation [1] by means of a bold MV-algebra on a singleton. However, this need not be more true for a general (real) chain with minimal and maximal element. Note that in such case, no MV-algebra may exist, or, if it exist, it need not be Archimedean.

Acknowledgment. The author is grateful to Prof. Mesiar for valuable discussions on t-norms. The support of the grant No. 95/5305/471 is kindly announced.

References

[1] L. Belluce, *Semisimple algebras of infinite-valued logic and Bold fuzzy set theory*, Canad. J. Math. **38** (1986), 1356 - 1379.
[2] C. C. Chang, *Algebraic analysis of many-valued logics*, Trans. Amer. Math. Soc. **88** (1958), 467 - 490.
[3] R. Cignoli, D. Mundici, *An invitation to Chang's MV-algebras*, in: M. Droste, R. Göbel, Advances in Algebra and Model Theory, OPA, Amsterdam (1997), 171 - 197.
[4] B. De Baets, R. Mesiar, *Triangular norm on product lattices*, Fuzzy Sets and Systems (in press).
[5] G. De Cooman, E. Kerre, *Order norms on bounded partially ordered sets*, J. Fuzzy Math. **2** (1994), 281 - 310.
[6] E. P. Klement, R. Mesiar, *Triangular norms*, Tatra Mt. Math. Publ. **13** (1997), 169 - 193.
[7] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, monograph in preparation.

[8] F. Kôpka, F. Chovanec, *D-posets*, Math. Slovaca **44** (1994), 21 - 34.

[9] C. H. Ling, *Representation of associative functions*, Publ. Math. Debrecen **12** (1965), 182 - 212.

[10] R. Mesiar, *Fuzzy sets, difference posets and MV-algebras*, in: B. Bouchon-Meunier, R. Yager, Fuzzy Logics and Soft Computing, World Scientific, Singapore (1995), 345 - 352.

[11] R. Mesiar, *Differences on [0,1]*, Tatra Mt. Math. Publ. **6** (1995), 131 - 140.

[12] R. Mesiar, *A note on extension of discrete triangular norms*, BUSEFAL (to appear).

[13] R. Mesiar, E. Pap, *Extension of differences on [0,1]*, Int. J. Theor. Phys. **33** (1994), 2317 - 2323.

[14] H. T. Nguyen, E. H. Walker, *A First Course in Fuzzy Logic*, CRC Press, Boca Ruten (1997).

[15] Z. Riečanová, D. Bršel, *Contraexamples in difference posets and orthoalgebras*, Int. J. Theor. Phys. **33** (1994), 133 - 141.

[16] B. Schweizer, A. Sklar, *Espaces Métrigues*, Aléatoires, Comptes Rendus Acad. Sci. 277, Paris (1958).

[17] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North Holland, New York (1983).

[18] E. Trillas, *Sobre funciones de negación en la teoriá de conjuntos diffusos*, Stochastica **3** (1979), 47 - 60.

Military Academy, Liptovský Mikuláš, Slovakia
E-mail address: rybarik@valm.sk