# Semantic of $R_*$-Dangerous Signal Recognition Logic Based on Boolean Algebra with Fuzzy Shell*

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### Abstract

In this paper, the new concept of Boolean algebras with Fuzzy shell is proposed, some important examples are given. A new implication operator, $R_*$-implication operator, and a new kind of valuation lattices, $R_*$-valuation lattices are made. And then, a new kind of nonclassical logic systems are established, the properties of this logic are investigated, some interesting results are obtained. Especially, it is discovered that in this logic, both 1-HS and 1-MP must hold unconditionally.

**Key Words**: Fuzzy logic; Boolean algebra with Fuzzy shell; Boolean heart; $R_*$-implication operator; $R_*$-valuation lattice; $\alpha$-tautology; $R_*$-Dangerous signal recognition logic; Approximate reasoning; Control principle.

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## 1. Introduction

A nonlinear ordered lattice (only six elements) has been used in dangerous signal recognition of circuit design successfully but few studies from mathematical structures and logical properties have been done. Taking those as the background, this paper deals with the generalization of this lattice in a more general sense. We'll first propose the new abstract concepts of Boolean algebra with Fuzzy shell, its Boolean

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heart, its Fuzzy shell. Then We'll give some examples of this kind of lattices. Secondly, a new implication operator, $R_*$-implication operator, is made, and a new valuation lattice, $R_*$-valuation lattice, is proposed. Thirdly, a new kind of nonclassical logic, which takes a $R_*$-valuation lattice as valuation lattice, and is called a $R_*$-dangerous signal recognition logic, is established. We'll give an elementary investigation of the logic system. Especially, We'll investigate the semantic of this logic and obtain some interesting results.

Our new logic system can be fractionized into two fragments, the Boolean heart, the Fuzzy shell. There will be two kind of logic structures with different and distinguished styles and features in our new system, a kind of Kleene-Dienes logic systems based on usual Boolean algebras, and another kind of Fuzzy logic systems which takes Kleene-Dienes operator as the implication operator and based on extensive Fuzzy lattices. Of cause, there exist many complicated situations in the investigation of transfragments, it is a interesting attractor in this logic systems. We'll discover that in the logic, both 1-HS and 1-MP must hold unconditionally.

## 2. Boolean algebra with Fuzzy shell

First, we are going to establish the new concept of Boolean algebra with Fuzzy shell, give some examples of this algebra, and discuss the special properties.

**Definition 2.1.** A distributive lattice $L$ is called *a Boolean algebra with Fuzzy shell*, if following conditions are satisfied:

(1) $L$ has the greatest element 1 and the least element 0.

(2) $L$ has an order-reversing involution $\neg$.

(3) $L$ has an unigue maximal Boolean type sublattice $L^{\#}$ such that the greatest element $1^{\#}$ and the least element $0^{\#}$ are different with 1 and 0 respectively, and the restriction of $\neg$ on $L^{\#}$ just coincide with the Boolean complement $'$ in $L^{\#}$.

$L^{\#}$ is called the *Boolean heart* of $L$, $\tilde{L}=L-L^{\#}$ is called the *Fuzzy shell* of $L$.

**Example 2.2.**

(1) Suppose that $B$ is any arbitrary Boolean algebra, $1^{\#}$ and $0^{\#}$ are the greatest element and the least element of $B$. Let $L=[0,\frac{1}{3}]\cup B\cup[\frac{2}{3},1]$, take usual ordering of real numbers in $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$. If $x\in[0,\frac{1}{3}], y\in B, z\in[\frac{2}{3},1]$, then

let $x<y<z$. $\forall\, y\in B$, let $\neg y=y'$; $\forall\, x\in[0,\frac{1}{3}]\cup[\frac{2}{3},1]$, let $\neg x=1-x$. Then $L$ is just a Boolean algebra with Fuzzy shell, where $L^{\#}=B$ is just the Boolean heart, and $\tilde{L}=[0,\frac{1}{3}]\cup[\frac{2}{3},1]$ is just the Fuzzy shell.

(2) Suppose that $B$ is any arbitrary Boolean algebra, $1^{\#}$ and $0^{\#}$are the greatest element and the least element of $B$ respectively. Take $1$, $0\notin B$, denote $L=B\cup\{0,1\}$. $\forall\, x\in B$, let $0<x<1$; $\forall\, x\in B$, let $\neg x=x'$; and let $\neg 0=1$, $\neg 1=0$. Then $L$ is just a Boolean algebra with Fuzzy shell, where $L^{\#}=B$ is just the Boolean heart, and $\tilde{L}=\{0,1\}$ is just the Fuzzy shell.

**Lemma 2. 3.** In any Boolean algebra with Fuzzy shell, de Morgan dual laws hold:

(1) $\neg(a\vee b)=\neg a\wedge\neg b$.

(2) $\neg(a\wedge b)=\neg a\vee\neg b$.

**Proposition 2. 4.** Suppose that $L$ is a Boolean algebra with Fuzzy shell, $L^{\#}$ and $\tilde{L}$ are the Boolean heart and the Fuzzy shell respectively, then

(1) For every $x\in L$, $\neg x\vee x\geqslant 1^{\#}$, $\neg x\wedge x\leqslant 0^{\#}$.

(2) For every $x\in L^{\#}$, $\neg x\vee x=1^{\#}$, $\neg x\wedge x=0^{\#}$

(3) For every $x\in\tilde{L}$, $\neg x\vee x\leqslant 1$, $\neg x\wedge x\geqslant 0$.

(4) $\neg 1=0$, $\neg 0=1$; $\neg 1^{\#}=0^{\#}$, $\neg 0^{\#}=1^{\#}$.

**Proposition 2. 5.** In any Boolean algebra $L$ with Fuzzy shell, does not exist any element $e$ such that

$$\neg e=e.$$

**Note 2. 6.** Any Boolean algebra is not a Boolean algebra with Fuzzy shell. Any Boolean algebra with Fuzzy shell is not a Boolean algebra. But the Boolean heart $L^{\#}$ of any Boolean algebra $L$ with fuzzy shell must be a Boolean algebra, its Fuzzy shell $\tilde{L}$ is a bounded distributive lattice with order-reversing involution. If we deal with the Boolean heart $L^{\#}$ and the Fuzzy shell $\tilde{L}$ of a Boolean algebra $L$ with Fuzzy shell separately, then they obey the corresponding operation laws respectively. But our more interest is just in the combined or fused investigation.

## 3. $R_*$-dangerous signal recognition logic

We are now going to establish another kind of nonclassical logic, $R_*$-dangerous signal recognition logic, which takes a Boolean algebra with Fuzzy shell as valuation lattice, and takes $R_*$-implication operator as implication operator.

**Definition 3. 1.** Suppose that $L$ is a Boolean algebra with Fuzzy shell. Let us make a mapping

$$R_*: L \times L \to L$$

as following

$$R_*(a,b)=\begin{cases}1, a \leqslant b, b \in \tilde{L}\\ 1^{\#}, a \leqslant b, b \in L^{\#}\\ \neg a \vee b, a \nleqslant b,\end{cases}$$

and call the mapping $R_*$ as *$R_*$-implication operator*. If we take $R_*$-implication operator $R_*$ as the implication operator$\to$ in the Boolean algebra $L$ with Fuzzy shell, then $L$ is called *a $R_*$-valuation lattice*. A mapping $v: F(S) \to L$ is called as a *$R_*$-valuation*, if $v$ is a homomorphism of type $(\neg, \vee, \wedge, \to)$. where F(S) is the free algebra of type $(\neg, \vee, \wedge, \to)$ generated by a nonempty set $S$. We denote the set of all $R_*$-valuations from F(S) to $L$ by $\Omega_{R_*}$.

**Definition 3. 2.** Suppose that $A \in F(S)$ and $a \in L-\{0\}$. If for every $R_*$-valuation $v \in \Omega_{R_*}$, $v(A) \geqslant a (v(A) > a, v(A) > 0, v(A) = 1, v(A) = 1^{\#})$, then the proposition $A$ is called an *$a$-tautology* (*$a^+$-tautology*, *pretautology*, *tautology*, *$\hat{1}^{\#}$-tautology*). We denote the set of all $a$-tautologies ($a^+$-tautologies, pretautologies, tautologies, $\hat{1}^{\#}$-tautologies) by $a\text{-}T(U)(a^+-T(U), QT(U), T(U), \hat{1}^{\#}-T(U))$.

**Definition 3. 3.** The heptalateral $\tilde{U}=(F(S), \Omega_{R_*}, a\text{-}T, a^+\text{-}T, QT, T, \hat{1}^{\#}\text{-}T)$ is called the *semantic of $R_*$-dangerous signal recognition logic $U$*.

**Definition 3. 4.** The ordered pair $U=(\tilde{U}, U=\hat{U})$ is called a *$R_*$-dangeroous signal recognition logic*, where $\hat{U}$ is the syntax of this logic.

**Proposition 3. 5.** For every family $\{a_t | t \in D\} \subset L-\{0\}$, we have

$$\bigcap_{i\in D}(a_i - T(U)) = (\bigcup_{i\in D} a_i) - T(U)).$$

**Note 3. 6.** In any $R_*$-valuation lattice $L$, we notice that $R_*$-implication operator $R_*$ doesn't coincide with wang Guojun implication operator

$$R_0 : L \times L \to L,$$

$$R_0(a,b) = \begin{cases} 1, & a \leqslant b, \\ \neg a \vee b, & a \nleqslant b, \end{cases}$$

because for each element $c$ of the Boolean heart $L^{\#}$,

$$R_*(c,c) = 1 \neq 1,$$

that is $R_*(c,c) \neq R_0(c,c)$.

**Proposition 3. 7.** In the Boolean heart $L^{\#}$ of a $R_*$-valuation lattice $L$, the restriction of $R_*$-implication operator $R_*$ onto $L^{\#}$ is just equivalent to Wang Guojun implication operator $R_0 : L^{\#} \times L^{\#} \to L^{\#}$,

$$R_0(a,b) = \begin{cases} 1^{\#}, & a \leqslant b, \\ \neg a \vee b, & a \nleqslant b, \end{cases}$$

and is also equivalent to Kleene-Dienes implication operator

$$R_{KD} : L^{\#} \times L^{\#} \to L^{\#},$$

$$R_{KD}(a,b) = \neg a \vee b.$$

**Proposition 3. 8.** In the Fuzzy shell $\tilde{L}$ of a $R_*$-valuation lattice $L$, the restriction of $R_*$-implication operator $R_*$ onto $\tilde{L}$ is just equivalent to Kleene-Dienes implication operator $R_{KD} : \tilde{L} \times \tilde{L} \to \tilde{L}$,

$$R_{KD}(a,b) = \neg a \vee b.$$

**Proposition 3. 9.** In any $R_*$-valuation lattice $L$, following revised Dubois-Prade conditions are satisfied:

(1) If $a \leqslant a^*$, then $R_*(a,b) \geqslant R_*(a^*,b)$.

(2) If $b \leqslant b^*$ and $a \nleqslant b^*$, then $R_*(a,b) \leqslant R_*(a,b^*)$.

(3) If $b \leqslant b^*$, and $a \leqslant b^*$, but $b^* \in \tilde{L}$,

then $\quad R_*(a,b) \leqslant R_*(a,b^*)$.

(4) $R_*(0,b) = \begin{cases} 1, & b \in \tilde{L}, \\ 1^{\#}, & b \in L^{\#}. \end{cases}$ And if $b \geqslant 0^{\#}$, then

$$R_*(0^{\#},b) = \begin{cases} 1, & b \in \tilde{L}, \\ 1^{\#}, & b \in L^{\#} \end{cases}$$

(5) $R_*(1,b)=b$. And if $b\geqslant 0^\#$, then

$$R_*(1^\#,b)=\begin{cases}1, & b>1^\#,\\ 1^\#, & b=1^\#,\\ b, & b<1^\#.\end{cases}$$

(6) $R_*(a,b)\geqslant b$.

(7) $R_*(a,a)=\begin{cases}1, & a\in\tilde{L},\\ 1^\#, & a\in L^\#.\end{cases}$

(8) $R_*(a,b)=1$ iff $a\leqslant b$ and $b\in L^\#$;

$R_*(a,b)=1^\#$ *iff* $a\leqslant b$ *and* $b\in\tilde{L}$.

**Note 3. 10.** Generally, $b\leqslant b^*\Rightarrow R_*(a,b)\leqslant R_*(a,b^*)$ in a $R_*$-valuation lattice $L$. For example, take $a$, $b$, $b^*\in L$ such that $a\leqslant b\leqslant b^*$, but $b\in\tilde{L}$ and $b^*\in L^\#$, then

$$R_*(a,b)=1>1^\#=R_*(a,b^*).$$

**Proposition 3. 11.** In the Boolean heart $L^\#$ of a $R_*$-valuation lattice $L$,

$$R_*(a,b)=R_*(\neg b,\neg a).$$

**Proposition 3. 12.** In the fuzzy shell $\tilde{L}$ of a $R_*$-valuation lattice $L$,

$$R_*(a,b)=R_*(\neg b,\neg a).$$

**Note 3. 13.** Generally, in a $R_*$- valuation lattice $L$,

$$R_*(a,b)\neq R_*(\neg b,\neg a).$$

For example, take $a\in L^\#$ and $b\in\tilde{L}$ such that $a\leqslant b$, then $\neg b\leqslant\neg a$, $\neg a\in L^\#$, and so

$$R_*(a,b)=1, R_*(\neg b,\neg a)=1^\#,$$

where $R_*(a,b)\neq R_*(\neg b,\neg a)$.

**Proposition 3. 14.** In any $R_*$-valuation lattice $L$,

$$R_*(a,R_*(b,a))\geqslant 1^\#.$$

**Proposition 3. 15.** In the Boolean heart $L^\#$ of a $R_*$-valuation lattice $L$,

$$R_*(a,R_*(b,a))=R_*(b,R_*(a,b))=1^\#;$$

in the Fuzzy shell $\tilde{L}$ of a $R_*$-valuation lattice $L$,

$$R_*(a,R_*(b,a)) = R_*(b,R_*(a,b)) = 1.$$

**Proposition 3. 16.** In the Boolean heart $L^{\#}$ of *a* $R_*$-valuation lattice $L$,

(1) $a \vee b = R_*(\neg a, b)$.

(2) $a \vee b = \neg(\neg a \wedge \neg b)$.

(3) $a \wedge b = \neg R_*(a, \neg b)$.

(3) $a \wedge b = \neg(\neg a \vee \neg b)$.

(5) $R_*(a,b) = \neg a \vee b$.

(5) $R_*(a,b) = \neg(a \wedge b \neg b)$.

**Proposition 3. 17.** In any $R_*$-valuation lattice $L$, if $a \leqslant b$ or $b \leqslant a$, then

$$R_*(a,b) \vee R_*(b,a) \geqslant 1^{\#}.$$

**Proposition 3. 18.** In the Boolean heart $L^{\#}$ of *a* $R_*$-valuation lattice $L$,

$$R_*(a,b) \vee R_*(b,a) = 1^{\#}.$$

**Proposition 3. 19.** In any $R_*$-valuation lattice $L$,

$$\neg a = R_*(a,0).$$

**Proposition 3. 20.** In any $R_*$-valuation lattice $L$,

(1) $b \wedge R_*(a,b) = b$.

(2) $b \vee R_*(a,b) = R_*(a,b)$.

(3) $R_*(a,R_*(a,b)) \geqslant R_*(a,b)$.

*If* $a \leqslant b$ *or* $a \not\leqq \neg a \vee b$, *then*

$$R_*(a,R_*(a,b)) = R_*(a,b).$$

*If* $a \not\leqq b$ *but* $a \leqslant \neg a \vee b$, *then*

$$R_*(a,R_*(a,b)) \geqslant 1^{\#}.$$

**Proposition 3. 21.** In the Boolean heart $L^{\#}$ of *a* $R_*$-valuation lattice $L$,

(1) $R_*(a,R_*(a,b)) = R_*(a,b)$.

(2) $\neg a \vee R_*(a,b) = R_*(a,b)$.

(3) $b \vee R_*(a,b) = R_*(a,R_*(a,b))$.

(4) $R_*(a,b \wedge c) = R_*(a,b) \wedge R_*(a,c)$.

(5) $R_*(a,b \vee c) = R_*(a,b) \vee R_*(a,c)$.

(6) $R_*(a \wedge b,c) = R_*(a,c) \vee R_*(b,c)$.

(7) $R_*(a \vee b,c) = R_*(a,c) \wedge R_*(b,c)$.

(8) $R_*(R_*(a,b),b)=a\vee b=R_*(\neg a,b)$.

(9) $R_*(R_*(a,b),b)=R_*(R_*(b,a),a)$.

(10) $R_*(a,\ R_*(b,c))=R_*(b,R_*(a,c))$.

**Proposition 3.22.** In the Boolean heart $L^\#$ of a $R_*$-valuation lattice $L$,

(1) $R_*(a,R_*(b,a))=1^\#$.

(2) $R_*(R_*(a,R_*(b,c)),R_*(R_*(a,b)),R_*(a,c)))=1^\#$.

(3) $R_*(R_*(\neg a,\neg b),R_*(b,a))=1^\#$.

(4) $R_*((R_*(a,b)\wedge a,b)=1^\#$.

(5) $R_*((R_*(a,b))\wedge\neg b,\neg a)=1^\#$.

(6) $R_*((a\vee b)\wedge\neg a,b)=1^\#$.

(7) $R_*(a\wedge b,a)=1^\#$.

(8) $R_*(a,a\vee b)=1^\#$.

(9) $R_*(R_*(a,\neg a),\neg a)=1^\#$,
$R_*(R_*(\neg a,a),a)=1^\#$.

(10) $(R_*(a,b))\wedge(R_*(R_*(b,c),R_*(a,c)))=1^\#$.

(11) $R_*(R_*(a,(c\wedge\neg c),\neg a)=1^\#$.

(12) $R_*(R_*(a,b),R_*(\neg b,\neg a))=1^\#$,
$R_*(R_*(\neg b,\neg a),R_*(a,b))=1^\#$.

(13) $R_*(R_*(a\wedge b,c),R_*(a,R_*(b,c)))=1^\#$.
$R_*(R_*(a,R_*(b,c)),R_*(a\wedge b,c))=1^\#$.

**Proposition 3.23.** In the Fuzzy shell $\widetilde{L}$ of a $R_*$-valuation lattice $L$,

(1) $R_*(a,R_*(b,a))=1$.

(2) $R_*(R_*(\neg a,\neg b),R_*(b,a))=1$.

(3) $R_*(R_*(a,R_*(b,c)),R_*(b,R_*(a,c)))=1$.

**Note 3.24.** A $R_*$-valuation lattice $L$ needn't be a Heyting algebra, because for $c\in L^\#$,

$$R_*(c,c)=1^\#\neq 1.$$

**Proposition 3.25.** In the Boolean heart $L^\#$ of a $R_*$-valuation lattice $L$,

(1) $a$—$HS$ *holds.*

(2) $a$—$MP$ *holds.*

**Proposition** 3. 26. In the Fuzzy shell $\tilde{L}$ of *a* $R_*$-valuation lattice $L$,

(1) 1—HS holds.

(2) 1—MP holds.

## 4. Conclusion

We have proposed the new concept, Boolean algebra with Fuzzy shell. Some important examples are given. We have made a new implication operator, $R_*$-implication operator, and a new valuation lattices, $R_*$-valuation lattices. Then a new kind of nonclassical logic systems, $R_*$-dangerous signal recognition logic, is established. $1^\#$, the greatest element of the Boolean heart, plays a special and important role in proposed logic.

Our new logic can be fractionized into two fragments, the Fuzzy shell, the Boolean heart. There are two kinds of logic structures with different and distinguished styles and features in our new logic, Fuzzy logic, and Kleene-Dienes logic, they are fused and combined each other by the characteristic algebraic structures of Boolean algebras with Fuzzy shell and $R_*$-implication operator. In the Boolean heart, many features and styles of Kleene-Dienes logic are shown or reappeared. In the Fuzzy shell, many features and styles of a kind of Fuzzy logic are shown indirectly and full. Of cause, there exist many complicated situations in the investigation of transfragments, it is another interesting attractor in this logic. We have discovered that 1-HS and 1-MP hold unconditionally in our logic.

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