THE FUZZY POWER AND FUZZY CARDINALITY OF FUZZY SETS

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Abstract In this paper we define the concepts of $\frac{1}{n}$-degree fuzzy power and fuzzy cardinality of fuzzy sets, generalize it to the T-degree fuzzy power and fuzzy cardinality of fuzzy sets and compare these definition ways with those appeared in other papers.

Key words fuzzy equipotence; Fuzzy power of fuzzy sets, Fuzzy cardinality

1. Preliminaries

Let X be a classical nonempty set and $\mathscr{P}(X)$ be power set of X and $\mathscr{F}(X)$ be the set of fuzzy subsets of X over $[0,1]$. We will denote by N the set of natural numbers, by I the interval $[0,1]$ and by I_i the interval $(\frac{i-1}{n}, \frac{i}{n}]$ for $1 \leqslant i \leqslant n, n \in N$. For $A \in \mathscr{F}(X)$ and $\lambda \in I$, let

$$A(n,i) = \{x \in X \mid \mu_A(x) \in I_i\}, \qquad A_{\bar{\lambda}} = \{x \in X \mid \mu_A(x) = \lambda\}$$

and call them I_i-point set and λ-point set of fuzzy set A respectively. Then, we have

$$A(n,i) = \bigcup_{\lambda \in I_i} A_{\bar{\lambda}} \qquad A_{\dot{0}} = \bigcup_{i=1}^{n} A(n,i)$$

the $A_{\dot{0}}$, therefore, is simply the support of fuzzy set A.

Let A and B be classical sets. We denote the power of A by $|A|$ and the class of cardinality by ***Card*** respectively. We write $A \approx B$ if they are equipotent.

2. Concepts of $\frac{1}{n}$-degree fuzzy power and fuzzy cardinality of fuzzy sets

Definition 2.1 Let A and B be fuzzy sets. A and B are called $\frac{1}{n}$-degree fuzzy equipotent if

$$A(n,i) \approx B(n,i)$$

for each i $(1 \leqslant i \leqslant n)$. Briefly, it is called $\frac{1}{n}$-degree equipotent and denoted by $A \overset{\frac{1}{n}}{\sim} B$.

For a given $n \in N$, the relationship $\overset{\frac{1}{n}}{\sim}$ is an equivalence relation. Then the calss of fuzzy sets can be partitioned into equivalence classes by $\overset{\frac{1}{n}}{\sim}$.

Definition 2.2 The $\frac{1}{n}$-degree equivalence class containing fuzzy set A is called the $\frac{1}{n}$-degree fuzzy power of A and denoted by $W_n(A)$.

Particularly, the $\frac{1}{n}$-degree equipotence relation is the same as the equipotence relation in classical set theory.

Proposition 2.3 Let A and B be fuzzy sets and $A \overset{\frac{1}{n}}{\sim} B$. If $m \mid n$, then $A \overset{\frac{1}{m}}{\sim} B$.

The proof is easy.

Let A, B be fuzzy sets and $n \in \mathbf{N}$. B is said to be $\frac{1}{n}$-degree more powered than A (denoted by $W_n(A) \leqslant W_n(B)$) if there exists an injection $f_i: A(n,i) \to B(n,i)$ for each $i(1 \leqslant i \leqslant n)$; B is said to be $\frac{1}{n}$-degree strictly more powered than A (denoted by $W_n(A) < W_n(B)$) if $W_n(A) \leqslant W_n(B)$ and there exists $i_0(1 \leqslant i_0 \leqslant n)$ such that $|A(n,i_0)| < |B(n,i_0)|$; We say that A is $\frac{1}{n}$-degree incomparable with B if there exist $i_1, i_2 (1 \leqslant i_1, i_2 \leqslant n, i_1 \neq i_2)$ such that $|A(n,i_1)| < |B(n,i_1)|$ and $|B(n,i_2)| < |A(n,i_2)|$.

Considering the $\frac{1}{n}$-degree fuzzy power, we can get that, for fuzzy sets A and B, exact one of the following results holds:

$W_n(A) = W_n(B)$; $W_n(A) < W_n(B)$; $W_n(B) < W_n(A)$; A is $\frac{1}{n}$-degree incomparable with B.

By virtue of the Cantor-Bernstein theorem in classical set theory, we can easily prove:

Theorem 2.4 (Cantor-Bernstein theorem for fuzzy sets)

If A and B are fuzzy sets with $W_n(A) \leqslant W_n(B)$ and $W_n(B) \leqslant W_n(A)$, then $W_n(A) = W_n(B)$.

Now, we give the concept of $\frac{1}{n}$-degree fuzzy cardinality of fuzzy sets.

Definition 2.5 For a fuzzy set $A \in \mathscr{F}(X)$, we say that A is a $\frac{1}{n}$-degree fuzzy cardinality if

(1) there exists $\kappa_i \in \mathbf{\mathit{Card}}$ for each i $(1 \leqslant i \leqslant n)$ such that $X = \bigcup_{i=1}^{n} \kappa_i \times \{i\}$;

(2) for $x \in \kappa_i \times \{i\}$, $\mu_A(x) = \frac{2i-1}{2n} (1 \leqslant i \leqslant n)$.

In the situation, we denote it by $A = FC(\kappa_1, \kappa_2, \cdots, \kappa_n)$.

Let B be a fuzzy set. The $\frac{1}{n}$-degree fuzzy cardinality which is $\frac{1}{n}$-degree equipotent to B is called the $\frac{1}{n}$-degree fuzzy cardinality of B and denoted by $Card_n(B)$.

Admitting the axiom of choice, we can give the only definite $\frac{1}{n}$-degree fuzzy cardinality $Card_n(B)$ for any given fuzzy set B. So Definition 2.5 is reasonable. We denote the class of $\frac{1}{n}$-degree fuzzy cardinality by $\mathscr{F}_c^n$.

Proposition 2.6 (1) If $A \in \mathscr{P}(X)$ and $A \approx \kappa$ (where κ is a classical cardinality), then $card_n(A) = FC(0, 0, \cdots, 0, \kappa)$.

(2) If $A \in \mathscr{F}(X)$ and $Card_n(A) = FC(\kappa_1, \kappa_2, \cdots, \kappa_n)$, then $A_0 \approx \bigcup_{i=1}^{n} \kappa_i \times \{i\}$.

(3) If $A, B \in \mathscr{F}_c^n$ and $A \overset{\frac{1}{n}}{\sim} B$, then $A = B$.

(4) Let A, B be fuzzy sets. Then $A \overset{\frac{1}{n}}{\sim} B$ iff $Card_n(A) = Card_n(B)$.

Let $A, B \in \mathscr{F}_c^n$, $A = FC(\kappa_1, \kappa_2, \cdots, \kappa_n)$, $B = FC(\lambda_1, \lambda_2, \cdots, \lambda_n)$. We order $\mathscr{F}_c^n$ by:

$A \leqslant B$ iff $\kappa_i \leqslant \lambda_i$ for each i $(1 \leqslant i \leqslant n)$;

$A < B$ iff $A \leqslant B$, moreover, there exists $i_0 (1 \leqslant i_0 \leqslant n)$ such that $\kappa_{i_0} < \lambda_{i_0}$;

A is incomparable with B iff there exist $i_1, i_2 (1 \leqslant i_1, i_2 \leqslant n, i_1 \neq i_2)$ such that $\kappa_{i_1} < \lambda_{i_1}$ and $\lambda_{i_2} < \kappa_{i_2}$.

From above, we have

Proposition 2.7 (1) $W_n(A) \leqslant W_n(B)$ iff $Card_n(A) \leqslant Card_n(B)$.

(2) $W_n(A) < W_n(B)$ iff $Card_n(A) < Card_n(B)$.

(3) $W_n(A)$ is incomparable with $W_n(B)$ iff $Card_n(A)$ is the same with $Card_n(B)$.

3. Generalization

In this section, we'll generalize the concepts of $\frac{1}{n}$-degree fuzzy power and fuzzy cardinality of fuzzy sets.

Definition 3.1 Let $\boldsymbol{T} \subset \mathscr{P}((0,1])$ and $\boldsymbol{T}$ is a cover of the interval $(0,1]$. $\boldsymbol{T}$ is called a left-open-right-closed exact cover (briefly denoted by *lec*) if each $J \in \boldsymbol{T}$ is a left-open-right-closed interval and $J \cap J' = \varnothing$ for distinct $J, J' \in \boldsymbol{T}$.

Definition 3.2 Let $A \in \mathscr{F}(X), B \in \mathscr{F}(Y)$. We say that A and B are $\boldsymbol{T}$-degree equipotent if $A_J \approx B_J$ for each $J \in \boldsymbol{T}$, where $\boldsymbol{T}$ is a *lec* of $(0,1]$ and $A_J = \{x \in X \mid \mu_A(x) \in J\}$, $B_J = \{y \in Y \mid \mu_B(y) \in J\}$.

Given a *lec* $\boldsymbol{T}$ of $(0,1]$, the $\boldsymbol{T}$-degree equipotence relation is an equivalence relation. Similarly, we can partition the class of fuzzy sets into equivalence classes by it.

Definition 3.3 The $\boldsymbol{T}$-degree equivalence class containing fuzzy set A is called the $\boldsymbol{T}$-degree fuzzy power of A and denoted by $W_{\boldsymbol{T}}(A)$.

Given a *lec* $\boldsymbol{T}$ of $(0,1]$, we denote by ${}^{\boldsymbol{T}}\boldsymbol{Card}$ the class of all the mappings from the collection $\boldsymbol{T}$ to *Card*.

Definition 3.4 $A \in \mathscr{F}(X)$ is called a $\boldsymbol{T}$-degree fuzzy cardinality if it satisfies:

(1) there exists $\alpha \in {}^{\boldsymbol{T}}\boldsymbol{Card}$ such that $X = \bigcup_{J \in \boldsymbol{T}} \alpha(J) \times \{J\}$;

(2) $A_J = \{x \in X \mid \mu_A(x) \in J\} = \alpha(J) \times \{J\}$;

(3) If $J = (x, y]$, then $\mu_A(a) = \frac{x+y}{2}$ for $a \in \alpha(J) \times \{J\}$.

The $\boldsymbol{T}$-degree fuzzy cardinality equipotent to fuzzy set B is called the $\boldsymbol{T}$-degree fuzzy cardinality of B.

There are many propositions of $\boldsymbol{T}$-degree fuzzy cardinality similar to those of the $\frac{1}{n}$-degree fuzzy cardinality and we will not discuss any more.

4. Comparison

"Power" and "cardinality" are two very important concepts in classical set theory. Concerning power and cardinality of fuzzy sets, some different definitions have been given in several literatures.

Let A be a fuzzy set and A_λ be the λ-level-set of A.

In [2], the following concept is given:

Definition 4.1 Let $A \in \mathscr{F}(X), B \in \mathscr{F}(Y)$. We say that A and B are equipotent if there exists a fuzzy bijection (i.e. $A_\lambda \approx B_\lambda$ for each $\lambda \in (0,1]$).

By the equivalence relation they partition the class of fuzzy sets into equivalence classes. The equivalence class containing fuzzy set A is called the power or cardinality of A.

In [3],Chen gives

Definition 4.2 Let A,B be fuzzy sets. A is said to be equipotent to B if $A_{\bar{\lambda}}\approx B_{\bar{\lambda}}$ for each $\lambda\in(0,1]$.

Definition 4.3 $A\in\mathscr{F}(N)$ is called a (countable) fuzzy cardinality if μ_A is monotone decreasing on N.

Then, we can get the following theorem easily:

Theorem 4.4 If $A_{\bar{\lambda}}\approx B_{\bar{\lambda}}$ for any $\lambda\in(0,1]$, then $A\overset{\frac{1}{n}}{\sim}B$ for any $n\in N$.

But its converse doesn't hold. This can be shown by the following example. Therefore, it is sure that Definition 2.1 is strictly weaker than Definition 4.2.

Example 4.5 Let $X=(0,+\infty)$ and $A,B\in\mathscr{F}(X)$. We define μ_A and μ_B as following:

$$\mu_A(x)=\begin{cases}\lambda & \text{if } x=n+\lambda \text{ where } n\in N,\lambda\in(0,1] \text{ and } \lambda\neq\frac{1}{2},\\ \frac{1}{2} & \text{if } x=\frac{1}{2},\\ 0 & \text{otherwise.}\end{cases}$$

$$\mu_B(x)=\begin{cases}\lambda & \text{if } x=n+\lambda \text{ where } n\in N,\lambda\in(0,1] \text{ and } \lambda\neq\frac{1}{2},\\ 0 & \text{otherwise.}\end{cases}$$

Obviously, $A\overset{\frac{1}{n}}{\sim}B$ for any $n\in N$ but $A_{\bar{\frac{1}{2}}}=\{1\}$ is not equipotent to $B_{\bar{\frac{1}{2}}}=\varnothing$.

Example 4.6 Let $A,B\in\mathscr{F}(N)$. Define μ_A and μ_B as following:

$$\mu_A(n)=\begin{cases}1 & \text{if } n=1,\\ 0.75 & \text{if } n \text{ is even},\\ 0.25 & \text{if } n \text{ is odd and } n\neq 1.\end{cases}$$

$$\mu_B(n)=\begin{cases}0.75 & \text{if } n \text{ is even},\\ 0.25 & \text{if } n \text{ is odd}.\end{cases}$$

Then $A\overset{\frac{1}{2}}{\sim}B$. But A it not equipotent to B in [2] since $A_1=\{1\}$ and $B_1=\varnothing$.

In the preceding example, A and B are almost the same except the value at one point $n=1$. In [2], A and B are not equipotent, so they belong to different equivalence classes. Considering infiniteness and ambiguity, we can know that the partition in [2] seems to be too crisp. And the condition of Definition 4.2 is even stronger than that of Definition 4.1. However, the definition of equipotence in this paper can avoid the case of Example 4.6 and characterize "power" of fuzzy sets with more fuzzy flavour.

In [1], the power of fuzzy set A is defined by $|A|=\sum_{i=1}^{n}\mu_A(x_i)$. It is limited to a discussion of those fuzzy sets with finite support sets. However, Definition 2.1 can be used to discuss both finite and infinte cases. Besides, we can choose different *lec*'s to meet different demands for ambiguity degree. So we can deal with different problems flexibly.

As for cardinality of fuzzy sets, we have to point out that Definition 4.3 is quite limited because there exist a large number of fuzzy sets with countable support sets for wich we could not find the corresponding fuzzy cardinality. For example: Let $X=[0,1]\cap Q$ and $A\in\mathscr{F}(X)$. Define

$\mu_A(x)=x$ for each $x \in X$. Then A is such a fuzzy set.

Comparing with others, the definition of fuzzy cardinality posed in this paper has following advantages:

(1) We can choose a fuzzy set as the corresponding fuzzy cardinality from the $\boldsymbol{T}$-degree equivalence class and the fuzzy cardinality can be conceived definitely.

(2) The way of the definition is in line with that of the definition of cardinality in classical set theory.

References

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